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THE CROSS-SECTION PAPER AS A MATHEMATICAL INSTRUMENT

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The following note[1] is addressed to teachers and prospective teachers of mathematics in elementary and secondary schools and colleges, who have come to recognize as fundamental the problem of closer *correlation of arithmetic, algebra, and geometry with one another and with the various domains of application*, or the problem of *unification of elementary pure and applied mathematics*. Pure mathematics is, as it were, a language for the convenient expression and investigation of relations the most diverse in ordinary life and in nature. The principles of the language are not arbitrary, but are imposed by the phenomena demanding convenient expression. The problem is solved when, and only when, we put our pupils into such a physical and intellectual environment that they learn to see and to think the mathematics for and of themselves.

In this note I wish to suggest possibilities, which may have escaped attention, in the systematic use of cross-section paper[2] as a unifying

[1] Read in part March 5, 1906, before a joint meeting of the Junior Mathematical Club and the Mathematical Club of the University High School of the University of Chicago.

[2] In the few diagrams to be given here only the square-ruled paper appears. I advise the use of the various styles of ruling—into squares, into rectangles, into parallelograms, into triangles, and with concentric circles and diverging radii—obtainable, for instance, from the Atlas School Supply Co., Chicago. The interaction of the various papers is especially important.

element in mathematics. I know of no medium serving to bring together so closely and so easily the three phases or dialects of pure mathematics—*number, form, formula*—and to lead so directly to the concept of *functionality*—a concept which since the seventeenth century has dominated advanced mathematics and the sciences; a concept which in the twentieth century, according to the auspices, will play a fundamental rôle in the reorganization of elementary mathematical education. *Functionality is the relation or (mathematical) law of connection between two or more quantities or numbers subject to simultaneous and interdependent continuous variation;* like, for instance, the relation between the simultaneous ages of a father and his son; or that between the length of the perimeter and the area of the surface of a square surrounded by a platoon of men preserving square formation and marching each directly away from a fixed flagpole; or that between the simultaneous temperature, pressure, and density of a gas under conditions varying in time or in space.

Students will gain a more easy and perfect mastery of mathematics, and their work will be full of richer direct and indirect value for them, when the primary emphasis is laid on the recognition, the depiction, and the closer study of functional relations between variable quantities. The variable quantity is the general quantity. I recognize the difficulties inherent in the notions of general and signed quantities and numbers, and in the notion of the depiction of a quantity of one kind to scale by a quantity of another kind. But in elementary teaching these difficulties should be met not primarily by the analytical processes of logic resulting in the separation of mathematics into arithmetic, algebra, geometry, only loosely related and not readily available for application. They should be met by the systematic union[1] of the three phases of mathematics in connection with problems

[1] The best modern textbooks on arithmetic show in a gratifying way that for the elementary school the problem of unification is under way to solution. The authors develop in organic relation to problems of real life, not only number work, but also to a considerable degree literal arithmetic, observational geometry, geometrical drawing, drawing to scale, plotting, and algebra.

Further, it is known that gratifying progress is making in various quarters toward the solution of the problem for the secondary school. Cf. the columns of the *School Review* and of *School Science and Mathematics*, Chicago, and J. W. A. Young's book, *The Teaching of Mathematics*, to be published soon by Longmans, Green & Co., for information as to the current movements in this country and in England, France, and Germany.

of practical and scientific nature. So approached, these fundamental conceptions are not really so difficult. Every boy loves trains, and most boys have traveled. With the cross-section paper in commonplace school use, the boy of nine or ten will readily understand and create diagrams of train motion, and will enjoy making the limited express overtake the slow freight at a certain time and place.

In this note, confining attention to the simplest algebraic and geometric aspects of functionality, I study the cross-section paper in connection with the questions of *double-entry tables* and *graphical computation*, and I hope to suggest in some measure the very central rôle which the cross-section paper as a mathematical instrument should be made to play in the unification of the elementary mathematical disciplines. By maximizing the function of the cross-section paper we secure, to speak only of pure mathematics, intense reaction between geometry and algebra. Geometry and algebra may certainly be developed independently, each with its relations to arithmetic, and no one doubts their high educational and scientific value as so developed. But this value is indeed small compared with the value to be obtained by developing them together in continuous reaction, thus releasing, as it were, abundant stores of sub-atomic energy.

The reader is requested to have at hand pencil, straight edge, and cross-section paper, and to duplicate the constructions shown in the figures, reserving for a second reading the more general parts of the text.

Double-Entry Tables—Numerical and Graphical

1. The fundamental operations of arithmetic and algebra involve two numbers, and the results of addition, etc., should be exhibited in double-entry tables, as indicated below, where for brevity the subtraction table is omitted and the given numbers X, Y are taken as positive integers at most 5. These tables, exhibiting the sum S, etc.,

$$X+Y=S\,,\quad XY=P\,,\quad \frac{Y}{X}=Q\,,$$

of the two numbers X, Y are to be thought of as inscribed on square-ruled paper, each entry S or P or Q being made at the corresponding point (X, Y) of the paper. Make such tables on large sheets with entries for X and Y varying, for instance, by fifths from -10 to $+10$,

and then see clearly how $S=X+Y$, $D=Y-X$, $P=XY$, $Q=Y/X$, or any function[1] F of X and Y as a function of position of the point (X, Y) varies (in general) continuously as the point (X, Y) moves continuously.

2. Evidently the numerical double-entry tables (I, II, III) may conveniently be replaced by the graphical tables[2] (Figs. 1, 2, 3);

Y \ X	1	2	3	4	5
5	6	7	8	9	10
4	5	6	7	8	9
3	4	5	6	7	8
2	3	4	5	6	7
1	2	3	4	5	6

I. Addition Table: $X+Y=S$

Y \ X	1	2	3	4	5
5	5	10	15	20	25
4	4	8	12	16	20
3	3	6	9	12	15
2	2	4	6	8	10
1	1	2	3	4	5

II. Multiplication Table: $XY=P$

Y \ X	1	2	3	4	5
5	5	$2\frac{1}{2}$	$1\frac{2}{3}$	$1\frac{1}{4}$	1
4	4	2	$1\frac{1}{3}$	1	$\frac{4}{5}$
3	3	$1\frac{1}{2}$	1	$\frac{3}{4}$	$\frac{3}{5}$
2	2	1	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{5}$
1	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$

III. Division Table: $\frac{Y}{X}=Q$

Y \ X	1	2	3	4	5
5	5.00	2.50	1.67	1.25	1.00
4	4.00	2.00	1.33	1.00	0.80
3	3.00	1.50	1.00	0.75	0.60
2	2.00	1.00	0.67	0.50	0.40
1	1.00	0.50	0.33	0.25	0.20

III. Division Table: $\frac{Y}{X}=Q$

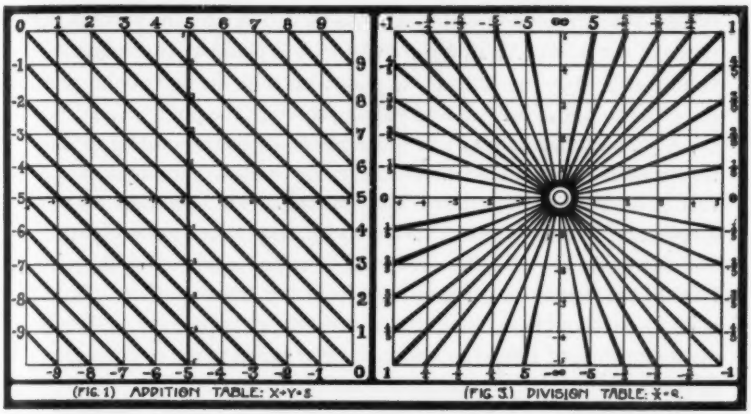
instead of making entries of the numerical values of S, etc., at individual points (X, Y) of the table, all the points (X, Y) with a certain

[1] A function F of two variable numbers X and Y is a variable number whose particular value for given values of X and Y is given or determinable in accordance with some table or formula or construction.

[2] The figures of this note are from drawings by Mr. J. Y. Lee, one of my pupils, to whom I express my gratitude.

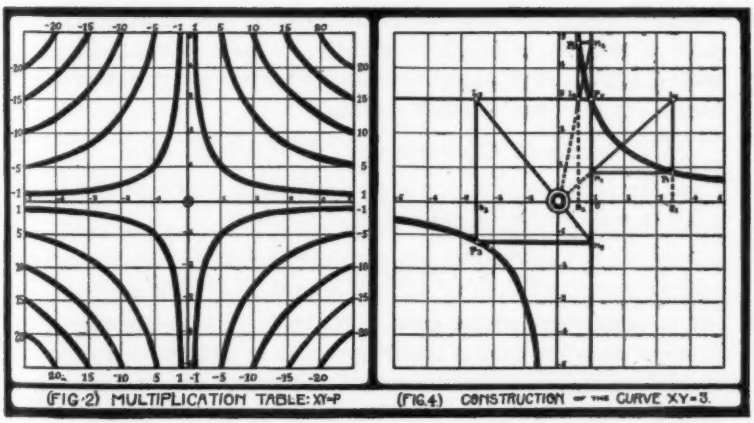
entry S are joined by a curve or straight line, which bears the proper entry once for all in the margin of the table.

3. Then one comes to the notion of a function of position in the plane—the Cartesian co-ordinates X, Y offering merely one mode of



(FIG. 1) ADDITION TABLE: X+Y=S (FIG. 3) DIVISION TABLE: $\frac{X}{Y}=Q$.

determining position, another important way being by the polar co-ordinates, the radius, and the angle of the circle-ruled paper—or even more generally of a function of position in space or on a curved



(FIG·2) MULTIPLICATION TABLE: XY=P (FIG.4) CONSTRUCTION OF THE CURVE XY=3.

surface. These notions, of course, enter at present, and are treated thus graphically especially in work with maps, whether geographical, topographical, or meteorological. We are merely to make mathematical use of these conceptions. Thus, if the graphical addition table (Fig. 1): $X+Y=S$, is taken as the topographic map of a countryside on the horizontal XY plane, S denoting the vertical distance of the point (X, Y, S) above the XY-plane as base-plane, the countryside is a flat hillside, a plane, cutting the horizontal plane $S=0$ in the line $X+Y=0$ and passing through the point $(X, Y, S)=(1, 2, 3)$. Models in clay and other materials corresponding to these topographic maps should be constructed; in classes in physiography this is often done by cutting and pasting. The question of topographic representation of a surface with respect to various (parallel and non-parallel) base-planes should be raised, and especially in the form of the derivation of one representation from another. In this connection should be used the triangle-ruled paper, its three axes diverging at 120° from the chosen origin being taken to correspond to the three mutually rectangular axes of space.

4. Of course, a table of single entry, say for Y as a function of X, is exhibited numerically by two columns of corresponding values of X, Y, and graphically as the single curve made up of the points (X, Y) whose co-ordinates correspond.

By the systematic use of the various kinds of cross-section paper for the construction and interpretation of graphical diagrams of data of whatever origin, it is possible to make our students see and enjoy a rich and suggestive variety of useful functional relations between two or three related variables.

Graphical Computations. Linkages A, A', B

5. In geometry it is customary by elementary processes to construct to scale arithmetic expressions of the form

$$3, \quad \frac{4}{5}, \quad \sqrt{5}, \quad 3 \pm 2\sqrt{5}, \quad \frac{2-3\sqrt{5}}{3-2\sqrt{5}}, \quad \ldots ;$$

that is, of numbers arising from 1 by the sequential application to numbers already at hand of the five processes of addition, subtraction, multiplication, division and extraction of square root of a positive number—and, more generally, of such expressions arising from one

or more numbers $a, b, \ldots$ supposed to be given to scale as linear segments. These constructions involve a large part of the theory of plane geometry and considerable use of the compasses. They are of limited significance, and hence relatively uninteresting.

But *introduce the element of functionality:*—for the variable number X construct and plot to scale the corresponding number Y, where Y denotes one of the various expressions in X,

$$X+3,\ 3X,\ -X+4,\ \frac{1}{X},\ X^2,\ \sqrt{X},\ \sqrt{1-X^2},\ \ldots,$$

and the situation changes immediately in significance and interest.

Use, however, the *constructional methods* of the square-ruled paper, for these methods are, in fact, exceedingly simple and powerful, and of far-reaching significance in theoretical and applied mathematics. Depending on the simplest properties of similar triangles and rectangles (cf. linkages A, A', B below), properties evident after a little experience with the paper, these constructional methods should be introduced in the grades in connection with drawing and observational geometry, and held in focus throughout the high-school and college course.

Indeed, for the purposes of elementary education *our current deductive geometry is of the nature of a fetish*, to be abandoned in favor of a geometry built on a richer system of geometric axioms—a system built to recognize the cross-section paper with its wealth of intuitional relations. In this wider environment there will be abundant need and opportunity for the systematic development of logical power. The matter of the elimination of redundant axioms should be touched on only lightly in secondary-school courses.

6. The cross-section paper is at present used primarily for *plotting*, that is, for recording in graphical form tables of arithmetical data, the result of *arithmetical computations* or of observations or of physical determinations. The constructional methods are in effect methods of *graphical computation*, and this constant checking of arithmetical and algebraic by geometric insight is the source of intense satisfaction, closely akin to that arising in the interaction between scientific theory and experimentation.

Arithmetical computations are so laborious that at present the graphical depiction of functional relations given by formula is of the

nature of a luxury, and I suppose everyone knows how little real significance attaches to the formula by itself. On the other hand, alert students and teachers will by various devices[1] execute with ease the graphical computations[2] of functional expressions.

7. For instruments use merely a straight edge in connection with the two systems of parallel linear scales furnished by the grating of the square-ruled paper, interpolating, if not by eye, by use of paper strip or dividers. But curves once graphically constructed are to be utilized as graphical tables; thus (*a*) the standard hyperbola $XY=1$ (Fig. 2) is a table of reciprocals Y of numbers X, and of reciprocals X of numbers Y; (*b*) the standard parabola $Y=X^2$ (Fig. 5) is a table of squares Y of numbers X, and of square roots X of numbers Y.

The graphical computation proceeds in scheme step by step parallel to the arithmetical computation. If a computation is impossible arithmetically—e. g., the extraction of the square root of a negative

[1] For instance, in Figs. 4 and 5 are indicated two constructions involving a straight line OLM, which appears in the various examples of each construction as OL_1M_1, OL_2M_2, OL_3M_3, etc. To effect these constructions expeditiously let the straight edge OLM of the ruler rest against a pin at O; then, in Fig. 5, since $UM=OB$, as M runs one by one along the marks of the scale UQ from U, the desired point L is marked on the consecutive vertical lines of the grating from the Y-axis, while, in Fig. 4, as L runs one by one along the marks of the line $Y_L=3$, with the eye one notes M on UP_0 and transfers by the eye, marking P on the vertical line through L.

Again, curves once constructed can be cut out and used later as rulers for the duplication of such curves perhaps in different positions with respect to the axes. The theory of such transformation of position of curves without change of form, and more generally the theory of transformations with or without change of form, and the corresponding theory of the transformation of the equations of the curves are of central importance.

Again, if a curve is known to be symmetrical with respect to some straight line or point, only one-half of the curve needs to be constructed directly.

Discriminate between the various curves of the same figure by letters for the general points of the respective curves (as below in this paper) and also graphically, by use of dots and dashes, shading, and colors, which should, of course, attach also to the elimination table and linkage diagram, which (as explained below) serve to epitomize the construction.

[2] I take the opportunity to refer to the article "Numerisches Rechnen," by Mehmke, in the *Encyklopädie der mathematischen Wissenschaften* (Leipzig: B. G. Teubner), Vol. I, article F, pp. 941–1081, for information as to the technical aspects of numerical computation, whether arithmetical, or by geometric drawing, or by the use of graphical tables (nomograms), or by apparatus, mechanical or physical. The nomographic methods are rapidly becoming of central importance. A nomogram once constructed for a certain type of problems of numerical computation, any particular

number—so is it graphically; cf. in §16 the construction for $\sqrt{1-x^2}$. If a computation is possible, but the result is not unique, as in the solution of equations, the arithmetical process usually requires tedious approximations, while the graphical process usually exhibits the results immediately in connection with the intersections of curves.

Of course, a computation of either type may be carried through in various ways; a comparison of these ways is interesting and instructive, and in the fundamental cases especially useful as serving to assist in determining a desirable set of geometric axioms.

8. In practice one handles addition directly by means of the linear scales of the paper. (Cf. §16.)

9. We propose the following problem in graphical multiplication. Designating in general the co-ordinates (X, Y) of a point K by X_K, Y_K, (to be read *the X of K, the Y of K*), having given a point P, we seek a point L such that to scale

$$X_L = X_P, \quad Y_L = X_P Y_P.$$

problem of the type is solved by the mere drawing of straight lines and noting the intersections of the lines with one another and with the curves of the nomogram.

Thus, to give a simple example, a nomogram for the solution of the general quadratic equation

$$X^2 + pX + q = 0$$

consists of the ordinary square-ruled paper with the standard parabola $Y = X^2$ (Fig. 5 below). The roots of any particular equation with given p, q are the X's of the points of section with the parabola of the straight line $Y = -pX - q$. This is the line crossing the X- and Y-axes in the respective points $(-\frac{q}{p}, 0)$, $(0, -q)$. One computes $\frac{q}{p}$ arithmetically or graphically (§9 below). Such an auxiliary computation is however unnecessary. Thus, d'Ocagne's nomogram (*loc. cit.*, §46) for the solution of

$$U^2 + pU + q = 0$$

consists merely of three curves (p), (q), (U) bearing three scales for p, q, U respectively, and the straight line joining the scale points p, q cuts the (U) curve in scale-points U, the roots of the equation to be solved. The three scales are, in terms of the square-ruled paper, as follows: the scale for p is the usual scale on the Y-axis; the scale for q is the usual scale on the line $X = 1$; the scale for U is $\left(\frac{1-X}{X}\right)$ on the hyperbola $Y = -\frac{(1-X)^2}{X}$.

Professor C. S. Slichter, of the University of Wisconsin, has arranged various styles of (uniform and non-uniform) ruling of cross-section paper (to be had of dealers in Madison) in such a way that all the elementary problems of logarithmic and trigonometric computation are to be solved merely by the drawing of straight lines.

This problem and naturally the corresponding problem in division: Given a point L, to determine the point P such that

$$X_P = X_L, \quad Y_P = \frac{Y_L}{X_L},$$

one solves by similar triangles using the figure $OUBLMP$, which appears in particular cases $OUB_1L_1M_1P_1$, $OUB_2L_2M_2P_2$, $OUB_3L_3M_3P_3$ in Fig. 4. We have in this figure in every case

$$X_M = OU = 1, \quad Y_M = UM = BP = Y_P, \quad X_L = OB = X_P,$$

and, from similarity of the triangles OUM, OBL,

$$\frac{OU}{OB} = \frac{UM}{BL} = \frac{BP}{BL},$$

that is,

$$\frac{1}{X_P} = \frac{1}{X_L} = \frac{Y_P}{Y_L},$$

so that indeed

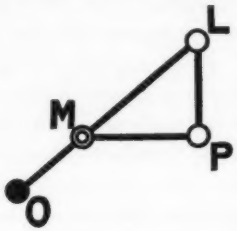
$$Y_P = \frac{Y_L}{X_L} \text{ and } Y_L = X_P Y_P.$$

It is to be noticed in the figure $OUB_3L_3M_3P_3$ how precisely the agreement or rule of signs in the multiplication of two signed numbers fits this geometric situation.

10. Geometric functionality comes to clearer vision by means of the notion of *linkage*—that is, an arrangement of links or bars with slots and pins, fixed or movable in slots, and pivots, and joints, and braces, such that, however the linkage be moved, certain essential properties of the diagram remain. The linkage represents geometrically the general variable diagram and algebraically a system of simultaneous equations between the simultaneous co-ordinates X, Y of the various points of the diagram, from which by elimination the relations sought are to be obtained.

In the *linkage diagrams* of this note points marked in black circles are fixed, those marked in white circles are free to move over the whole plane, those marked in two concentric circles are free to move each along a prescribed line or curve in the plane, while straight lines shown may move but so as to remain straight, and straight lines shown in the X- and Y-directions may move but so as to retain those directions.

Thus the linkage diagram and the ***elimination table*** for the multiplication figure discussed above are as follows:



$O=(0, 0)$

$X_M=1$

IV. **LINKAGE A**

	L	M	P
L		$\frac{Y_L}{X_L}=\frac{Y_M}{X_M}$ $\therefore Y_L=X_L Y_M$	$X_L=X_P$ $\therefore Y_L=X_P Y_P$ $\therefore Y_P=\frac{Y_L}{X_L}$
M	——	$X_M=1$	$Y_M=Y_P$
P	——	——	

IV. ELIMINATION TABLE

An elimination table for a diagram involving the variable points L, M, P is an arrangement of six compartments LL, LM, LP, MM, MP, PP, where a compartment—e. g., MP—contains the entries connecting the points M and P, here the entry $Y_M=Y_P$. Here the entries preceded by the sign ($\therefore$) for "therefore" are obtained by elimination from the other four entries, which correspond directly to the conditions of the diagram. The entry MM, $X_M=1$, specifies the curve (M), in this case a line, on which M is to move. There are no entries LL, PP, because the points L, P are individually unconditioned and free to move over the whole plane.

11. Linkage A is a graphical multiplying machine: $Y_L = X_P Y_P$, and a graphical dividing machine: $Y_P = \frac{Y_L}{X_L}$. In any case of multiplication or division, if the two given numbers are not thus given graphically to scale as the co-ordinates (X, Y) of a point P or L, then a preliminary transformation of the data or a suitable modification of the process becomes necessary. Such a transformation would be by means of the lines of the grating and the standard diagonal $Y = X$, (OQ of Fig. 5 below) which throughout plays the rôle of a *turner* of X-segments into equal Y-segments, and *vice versa*.

For instance, if two points H and K are given, with $X_H = X_K$, the point G with

$$X_G = X_H = X_K, \quad Y_G = Y_H Y_K$$

is constructed by means of the figure involving linkage (A) and the turner (T), $Y_T = X_T$, specified by the following relations:

$$Y_H = Y_T = X_T = X_P, \quad Y_K = Y_P, \quad Y_L = Y_G, \quad X_G = X_H = X_K.$$

Construct linkage diagram and elimination table for this figure.

12. Clearly, if in linkage A the point P be made to move along some definite curve, call it the curve (P), the locus of the point P, corresponding to the algebraic relation[1]

$$(P): \; Y_P = \mathbf{P}(X_P); \quad \text{for instance, } Y_P = X_P,$$

between the co-ordinates X_p, Y_p, where $\mathbf{P}(X_p)$, (*the P function of the X of P*) denotes in general the algebraic expression in terms of

[1]As a result of the law of the motion of the point P, its co-ordinates X_P, Y_P will be always related in a certain way, expressed algebraically by the fact that Y_P always represents to scale a certain algebraic expression or function $\mathbf{P}(X_P)$ in terms of the co-ordinate X_P as taken to scale. We write for the curve (P),

$$(P): \; Y_P = \mathbf{P}(X_P),$$

or as another expression of the same meaning,

$$(P): \; X_P = \mathbf{P}^{-1}(Y_P).$$

These two equations are read: *the Y of P equals the P function of the X of P; the X of P equals the inverse P function of the Y of P.*

Every curve (P) determines in this way by the functional relations between the co-ordinates X, Y of its variable point P a pair of mutually inverse functions:

$$\mathbf{P}(Z), \quad \mathbf{P}^{-1}(Z),$$

and conversely every pair of inverse functions is representable in this way by a curve drawn to scale with respect to a co-ordinate system. The notations used are intended

X_P which is to scale equal to Y_P for every point P of the curve (P), then the point L will move along some curve (L),

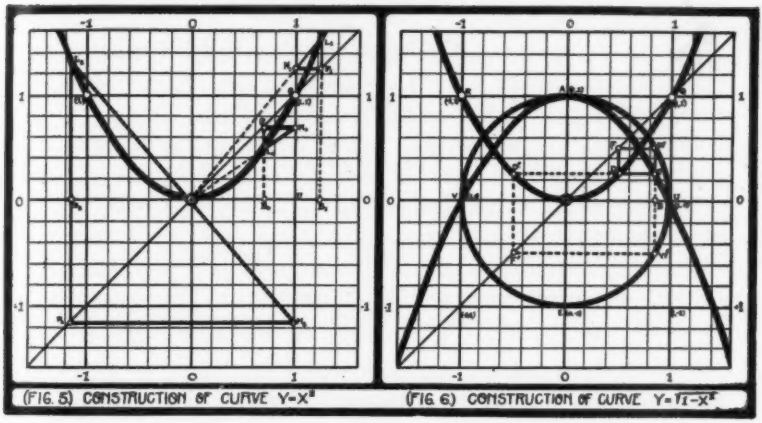
$$(L):\ Y_L=\mathbf{L}(X_L)=X_L\cdot\mathbf{P}(X_L)\ ;\ \text{in particular, } Y_L=X_L^2\ .$$

Similarly, if L move along (L),

$$(L):\ Y_L=\mathbf{L}(X_L)\ ;\ \text{for instance, } Y_L=3\ ,$$

then P will move along (P),

$$(P):\ Y_P=\frac{\mathbf{L}(X_P)}{X_P}\ ;\ \text{in particular, } Y_P=\frac{3}{X_P}\ .$$



(FIG. 5) CONSTRUCTION OF CURVE $Y=X^2$ (FIG. 6) CONSTRUCTION OF CURVE $Y=Y_1-X^2$

Thus, in particular, we obtain the graphical construction of (Fig. 5) the standard parabola[1] (L): $Y_L=X_L^2$, and (Fig. 4) the rec-

to emphasize the essential equivalence of the curve and the pair of inverse functions.

Given a curve (P) and a point Q; let the vertical line through the point Q meet the curve (P) in the point P_1; and let the horizontal line through the point Q meet the curve (P) in the point P_2. Then $X_Q=X_{P_1}$ and $Y_Q=Y_{P_2}$ so that

$$Y_{P_1}=\mathbf{P}(X_Q)\ ,\quad X_{P_2}=\mathbf{P}^{-1}(Y_Q)\ ,$$

that is, *the Y of P_1 is the P function of the X of Q; the X of P_2 is the inverse P function of the Y of Q.*

Show how by use of the turner to compute graphically from the curve (P) and the point Q the functional expressions

$$\mathbf{P}(Y_Q)\ ,\quad \mathbf{P}^{-1}(X_Q)\ .$$

[1]Students of projective geometry will notice that the metrical constructions of Figs. 4 and 5 are precisely the Pascal hexagon projective constructions for conics,

tangular hyperbola (P): $X_P Y_P = 3$, with the following elimination tables and linkage diagrams:

V. **LINKAGE A'** (FIGS 5 AND 4)

$O=(0, 0)$	$O=(0, 0)$
$X_M=1$	$X_M=1$
$Y_P=X_P$	$Y_L=3$
$\therefore Y_L=X_L^2$	$\therefore X_P Y_P=3$
$(V$, Fig. 5)	$(V$, Fig. 4)

	L	M	P
L	$\therefore Y_L=X_L^2$	$Y_L X_M=X_L Y_M$	$X_L=X_P$
M	——	$X_M=1$	$Y_M=Y_P$
P	——	——	$Y_P=X_P$

(V, FIG. 5) PARABOLA $Y_L=X_L^2$

	L	M	P
L	$Y_L=3$	$Y_L=X_L Y_M$	$X_L=X_P$
M	——	$X_M=1$	$Y_M=Y_P$
P	——	——	$\therefore X_P Y_P=3$

(V, FIG. 4) HYPERBOLA $X_P Y_P=3$

being given three points and tangent lines at two of the three points, viz. (Fig. 4), the point P_0 and the X- and Y-axes as asymptotes; (Fig. 5) the point Q and the X-axis and the line at infinity as tangents at their respective intersections with the Y-axis.

13. $\mathbf{L}(Z)=Z\cdot\mathbf{P}(Z)$.—This linkage A' yields, as stated above, more general curves for P, L, respectively, if L, P are made to traverse more general curves, viz.:

V. **LINKAGE A' (GENERAL)** $\mathbf{L}(Z)=Z\cdot\mathbf{P}(Z)$

$$O=(0,0) \qquad O=(0,0)$$
$$X_M=1 \qquad X_M=1$$
$$Y_P=\mathbf{P}(X_P) \qquad Y_L=\mathbf{L}(X_L)$$
$$\therefore Y_L=X_L\cdot\mathbf{P}(X_L)=\mathbf{L}(X_L) \qquad \therefore Y_P=\frac{\mathbf{L}(X_P)}{X_P}=\mathbf{P}(X_P)$$

For instance, if P traverses the quadratic parabola $Y=X^2$ of Fig. 5, L traverses the cubic parabola $Y=X^3$; if P traverses this parabola, L traverses the quartic parabola $Y=X^4$. Thus, by applying linkage A', being given the curve (P) and finding the curve (L), in succession starting from the standard diagonal $Y=X$ we obtain the curves $Y=X^2$, $Y=X^3$, $Y=X^4$, ... and, in general, $Y=X^n$ for n a positive integer.[1] The first quadrant of this curve serves for the whole curve, since the curve is symmetrical with respect to the Y-axis (n even) or the origin (n odd), and, it serves also, by interchanging the rôles of the X- and Y-axes, for the curve $X=Y^n$, that is, for the curve $Y=X^{\frac{1}{n}}$.

14. $\mathbf{G}(Z)=\mathbf{H}(Z)\cdot\mathbf{K}(Z)$.—Similarly, by means of the linkage of §11, we construct from two given curves (H), (K):

$$Y_H=\mathbf{H}(X_H)\,, \quad Y_K=\mathbf{K}(X_K)\,,$$

the curve (G):

$$Y_G=\mathbf{H}(X_G)\cdot\mathbf{K}(X_G)\,,$$

viz., *the product as to Y of the two curves (H) (K), in the sense*

$$X_G=X_H=X_K\,, \quad Y_G=Y_HY_K\,,$$

so that

$$\mathbf{G}(Z)=\mathbf{H}(Z)\cdot\mathbf{K}(Z)\,,$$

where Z denotes an arbitrary number the basis of the functional expressions **G**, **H**, **K**.

Of course, the same linkage otherwise employed yields from the curves (G) and (H) the curve (K) as the quotient as to Y of (G) by (H).

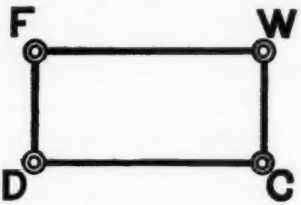
[1] The general case, n fractional, positive or negative, is treated in §21.

15. The linkage of §11 specializes to linkage A if $H=T$, and accordingly $K=P$, $G=L$. We have (cf. §13)

$$X_T=X_P=X_L\,,\quad Y_L=X_PY_P=Y_TY_P\,,$$

so that the curve (L) is the product as to Y of the curve (P) and the turner (T).

16. The rectangular linkage B (VI) is of central importance in the compounding of curves and functions. Three points C, D, F traverse given curves (C), (D), (F), and then W traverses a certain curve (W).



$$Y_W=Y_F\,,\quad X_F=X_D\,,$$
$$Y_D=Y_C\,,\quad X_C=X_W\,.$$

VI. LINKAGE B

Thus, if D traverses the turner $Y_D=X_D$ and F the straight line $Y_F=3\cdot X_F$ and C a curve $Y_C=\mathbf{C}(X_C)$, then

$$Y_W=Y_F=3\cdot X_F=3\cdot X_D=3\cdot Y_D=3\cdot Y_C=3\cdot\mathbf{C}(X_C)=3\cdot\mathbf{C}(X_W),$$

so that this use of linkage B effects the transformation of the curve (C) into a curve (W) as follows:

$$X_W=X_C\,,\quad Y_W=3\cdot Y_W\,,$$

and correspondingly constructs from a function $\mathbf{C}(Z)$ a function $\mathbf{W}(Z)$ where

$$\mathbf{W}(Z)=3\cdot\mathbf{C}(Z)\quad.$$

Similarly, arrange the linkage to construct from a function $\mathbf{P}(Z)$, graphically given by a curve $(\mathbf{P})$: $Y_P=\mathbf{P}(X_P)$, the various functions,

$$\mathbf{W}(Z)=2\cdot\mathbf{P}(Z)\,,\ \tfrac{1}{2}\mathbf{P}(Z)\,,\ \mathbf{P}(2Z)\,,\ \mathbf{P}(\tfrac{1}{2}Z)\,,$$

and

$$\mathbf{W}(Z)=\mathbf{P}(Z)+2\,,\ \mathbf{P}(Z)-2\,,\ \mathbf{P}(Z+2)\,,\ \mathbf{P}(Z-2)\quad.$$

In these cases D traverses the turner and either C or F the curve (P), while the third point F or C traverses a straight line.

17. To apply linkage B to construct (Fig. 6) the circle

$$Y_W=\sqrt{1-X_W^2}$$

of radius 1 and center at origin, let F traverse the turner $Y=X$, and D the standard parabola $Y=X^2$, $X=\sqrt{Y}$, and C the parabola[1] $Y=1-X^2$, that is,

$$Y_F=X_F\,,\quad X_D=\sqrt{Y_D}\,,\quad Y_C=1-X_C^2\,,$$

so that indeed

$$Y_W=Y_F=X_F=X_D=\sqrt{Y_D}=\sqrt{Y_C}=\sqrt{1-X_C^2}=\sqrt{1-X_W^2}\,.$$

Construct this circle

$$Y_W=\sqrt{1-X_W^2}\,,\quad X_W^2+Y_W^2=1$$

by the same linkage also as follows:

$$X_D+Y_D=1\,,\quad Y_C=X_C^2\,,\quad X_F=Y_F^2\,;$$

so that indeed

$$1=X_D+Y_D=X_F+Y_C=Y_F^2+X_C^2=Y_W^2+X_W^2\,.$$

The curve (D) is in this case the straight line AU of Fig. 6.

18. $\mathbf{W}(Z)=\mathbf{F}\mathbf{D}^{-1}\mathbf{C}(Z)$.[2]—The curve (W) is in a certain sense compounded out of the curves (C), (D), (F). Just so the two mutually inverse functions $\mathbf{W}(Z)$, $\mathbf{W}^{-1}(Z)$ associated with the curve (W) are compounded out of the functions associated with the curves (C), (D), (F). For, in the customary notations, with

$$Y_W=\mathbf{W}(X_W),$$

we have

$$\begin{aligned}Y_W=Y_F&=\mathbf{F}(X_F)=\mathbf{F}(X_D)=\mathbf{F}\big(\mathbf{D}^{-1}(Y_D)\big)\\&=\mathbf{F}\big(\mathbf{D}^{-1}(Y_C)\big)=\mathbf{F}\Big(\mathbf{D}^{-1}\big(\mathbf{C}(X_C)\big)\Big)\\&=\mathbf{F}\Big(\mathbf{D}^{-1}\big(\mathbf{C}(X_W)\big)\Big)\,,\end{aligned}$$

that is, in briefer form,

$$Y_W=\mathbf{W}(X_W)=\mathbf{F}\mathbf{D}^{-1}\mathbf{C}(X_W)\,,$$

so that as operator on a general number Z the function $\mathbf{W}$ is the

[1] Obtained from $Y=X^2$ by reflection on X-axis and by lifting 1 in the direction of the Y-axis. Make this curve $Y=1-X^2$ also by use of linkage B as follows:

$$Y_F=1-X_F\,,\quad X_D=Y_D\,,\quad Y_C=X_C^2\,,$$

so that

$$Y_W=Y_F=1-X_F=1-X_D=1-Y_D=1-Y_C=1-X_C^2=1-X_W^2\,.$$

[2] To be read, the W function of Z equals the F function of the inverse D function of the C function of Z. This is the composition of functions effected in general by linkage B.

function resulting from the composition of the functions $\mathbf{C}$, $\mathbf{D}^{-1}$, $\mathbf{F}$ in that order, viz.

(*a*) $$\mathbf{W}(Z)=\mathbf{F}\mathbf{D}^{-1}\mathbf{C}(Z)\,.$$

Similarly,

(*b*) $$\mathbf{W}^{-1}(Z)=\mathbf{C}^{-1}\mathbf{D}\mathbf{F}^{-1}(Z)\,.$$

The formula (*a*) was just now derived from the linkage relations between the curves (*C*), (*D*), (*F*), (*W*) by process of algebraic elimination to express Y_W in terms of X_W.

It is desirable to see conversely how the linkage enables us to compute graphically by means of (*C*), (*D*), (*F*) for every number Z a corresponding number $\mathbf{W}(Z)=\mathbf{F}\,\mathbf{D}^{-1}\,\mathbf{C}(Z)$. Taking the Z as an X, say X_C, we have $\mathbf{C}(Z)$ given as

$$\mathbf{C}(Z)=\mathbf{C}(X_C)=Y_C\,,$$

and then

$$\mathbf{D}^{-1}\mathbf{C}(Z)=\mathbf{D}^{-1}(Y_C)=\mathbf{D}^{-1}(Y_D)=X_D\,,$$

and then

$$\mathbf{W}(Z)=\mathbf{F}\mathbf{D}^{-1}\mathbf{C}(Z)=\mathbf{F}(X_D)=\mathbf{F}(X_F)=Y_F\,.$$

Thus, to express Y_W in terms of X_W in the form $Y_W=\mathbf{F}\mathbf{D}^{-1}\mathbf{C}(X_W)$, we proceed around the rectangle in the sense *W F D C W*, and conversely, to compute graphically from given X_W the expression $\mathbf{F}\mathbf{D}^{-1}\mathbf{C}(X_W)$ as Y_W we proceed around the rectangle in the opposite sense *W C D F W*.

19. $\mathbf{W}(Z)=\mathbf{D}^{-1}(Z)$.—If *C* and *F* traverse the turner $Y=X$, then *D* and *W* are always symmetrical with respect to the turner, and

$$X_W=X_C=Y_C=Y_D\,,\quad Y_W=Y_F=X_F=X_D\,,$$

so that linkage *B* transforms the curve (*D*),

$$(D):\ Y_D=\mathbf{D}(X_D)\,,$$

into the curve (*W*) symmetrical to (*D*) with respect to the turner, viz.:

$$(W):\ X_W=\mathbf{D}(Y_W)\,,\ \text{i. e.,}\ Y_W=\mathbf{D}^{-1}(X_W)\,.$$

Here

$$\mathbf{W}(Z)=\mathbf{D}^{-1}(Z)\,,$$

so that this is a construction for the function $\mathbf{W}(Z)$ inverse to a given function $\mathbf{D}(Z)$.

The present relation $\mathbf{W}(Z)=\mathbf{D}^{-1}(Z)$ is in accord with the general

relation $\mathbf{W}(Z)=\mathbf{F}\,\mathbf{D}^{-1}\,\mathbf{C}(Z)$ of §17, since in the present case for every number Z $\mathbf{F}(Z)=Z$ and $\mathbf{C}(Z)=Z$.

20. $\mathbf{W}(Z)=\mathbf{FC}(Z)$ or $\mathbf{FD}^{-1}(Z)$ or $\mathbf{D}^{-1}\mathbf{C}(Z)$.—If D traverses the turner, so that $\mathbf{D}^{-1}(Z)=Z$, then linkage B constructs from given functions $\mathbf{C}(Z)$, $\mathbf{F}(Z)$ the compound function

$$\mathbf{W}(Z)=\mathbf{FC}(Z)\,.$$

Similarly, if C or F traverses the turner, we construct the compound function

$$\mathbf{W}(Z)=\mathbf{FD}^{-1}(Z) \text{ or } \mathbf{D}^{-1}\mathbf{C}(Z)\,.$$

21. The curve $Y=X^k$, associated with the function Z^k where $k=\frac{n}{d}$, n and d positive relatively prime integers, is constructed by linkage B from the curves:

$$(C)\quad Y_C=X_C^n\,;\qquad \mathbf{C}(Z)=Z^n\,,$$

$$(D)\quad Y_D=X_D^d\,;\quad \mathbf{D}^{-1}(Z)=\frac{1}{Z^d}\,,$$

which were constructed in §13, while F traverses the turner, for by §20

$$\mathbf{W}(Z)=\mathbf{D}^{-1}\mathbf{C}(Z)=(Z^n)^{\frac{1}{d}}=Z^{\frac{n}{d}}=Z^k\,,\quad Y_W=\mathbf{W}(X_W)=X_W^k\,.$$

The curve $Y=X^{-k}=X^{-\frac{n}{d}}$ may be constructed from the same curves (C) (D) by letting F traverse the standard rectangular hyperbola

$$(F):\ Y_F=\frac{1}{X_F}\,;\quad \mathbf{F}(Z)=\frac{1}{Z}\,,$$

(cf. §12, Fig. 4), for

$$\mathbf{W}(Z)=\mathbf{FD}^{-1}\mathbf{C}(Z)=\frac{1}{Z^{\frac{n}{d}}}=Z^{-\frac{n}{d}}=Z^{-k}\,,$$

$$Y_W=\mathbf{W}(X_W)=X_W^{-k}\,.$$

Construct it also thus:

$$\mathbf{F}(Z)=\frac{1}{Z}\,,\quad \mathbf{D}(Z)=Z\,,\quad \mathbf{C}(Z)=Z^k\,;$$

so that indeed

$$\mathbf{W}(Z)=\mathbf{FC}(Z)=\frac{1}{Z^k}=Z^{-k}\,.$$

22. The system of curves $Y=X^k$, $k=\pm\frac{n}{d}$, is worthy of remark. All the curves contain the point (1, 1) and run through the first quadrant $X>0$, $Y>0$ of the plane. We consider their distribution in the first quadrant. Those with k positive contain the origin (0, 0), those with k negative are asymptotic to both the co-ordinate axes. These two classes $(k=+)$, $(k=-)$ are separated by the curves $(k=0)$: $Y=1$, $(k=\infty)$: $X=1$, which partition the quadrant into four (call them) quarters; the class $(k=+)$ lies in the SW and NE quarters (to speak geographically as from the point (1, 1)), while the class $(k=-)$ lies in the SE and NW quarters of the quadrant. The curves $(k=\pm)$ fall into two classes $(k=\pm>)$, $(k=\pm<)$ according as k is numerically greater than or less than 1. The turner $(k=+1)$: $Y=X$ subdivides the class $(k=+)$, while the standard hyperbola $(k=-1)$: $Y=X^{-1}$ subdivides the class $(k=-)$. There are thus eight octants of the first quadrant. The curves $(k=+>)$ occupy the SSW and NNE octants, and come into the origin tangentially to the X-axis; the curves $(k=+<)$ occupy the WSW and ENE octants, and come into the origin tangentially to the Y-axis. The area inclosed by the turner, either co-ordinate axis, and the hyperbola $(k=-1)$ is in fact infinite. The curves $(k=-<)$ occupy the WNW and ESE octants, and inclose with the Y-axis a finite, and with the X-axis an infinite, area. The curves $(k=->)$ occupy the NNW and SSE octants, and inclose with the Y-axis an infinite, and with the X-axis a finite, area. These statements concerning areas are capable of elementary proof. The curves $Y_A=X_A^k, Y_B=X_B^{-k}$ are (cf. §14) as to Y reciprocals each of the other, since $X_A=X_B$ implies $Y_AY_B=1$; and they are like wise as to X reciprocals each of the other, since $Y_A=Y_B$ implies $X_AX_B=1$.

The march of the curves $Y=X^k$ in the other three quadrants of the plane depends upon the evenness—oddness character of the numerator and denominator of $k=\pm\frac{n}{d}$. The curves $\left(k=\frac{\text{even}}{\text{odd}}, \frac{\text{odd}}{\text{odd}}, \frac{\text{odd}}{\text{even}}\right)$ are symmetrical respectively as to the Y-axis, the origin, the X-axis, and pass from the first quadrant into the second, the third, the fourth quadrant.

23. By use of the curves $Y=X^k, Y=X^l$, each of the constructions (§17) of the circle by linkage B may be generalized, giving the curve

$$Y=\sqrt[l]{1-X^k}\,,\quad X^k+Y^l=1\,,$$

viz., in the first place,

$$Y_W=Y_F=X_F=X_D=\sqrt{Y_D}=\sqrt[l]{Y_C}=\sqrt[l]{1-X_C^k}=\sqrt[l]{1-X_W^k}\,,$$

and, in the second place,

$$1=X_D+Y_D=X_F+Y_C=Y_F^l+X_C^k=Y_W^l+X_W^k\,.$$

The second construction is the more symmetrical. In it the curve $X_W^k+Y_W^l=1$ arises by transformation via the curves

$$Y_C=X_C^k\,,\quad X_F=Y_F^l$$

out of the straight line (AU of Fig. 6)

$$Y_D+X_D=1\,.$$

24. More generally, any curve

$$\mathbf{C}(X_W)+\mathbf{E}(Y_W)=1$$

arises by transformation via the curves

$$Y_C=\mathbf{C}(X_C)\,,\quad X_F=\mathbf{E}(Y_F)=\mathbf{F}^{-1}(Y_F)$$

out of the straight line

$$Y_D+X_D=1\,;$$

and similarly, in general, any curve

$$\Delta\big(\mathbf{C}(X_W)\,,\ \mathbf{E}(Y_W)\big)=0$$

is constructed from the curve

$$\Delta(Y_D\,,\ X_D)=0$$

by means of the curves

$$Y_C=\mathbf{C}(X_C)\,,\quad X_F=\mathbf{E}(Y_F)\,,$$

viz.;

$$0=\Delta(Y_D,\ X_D)=\Delta(Y_C,\ X_F)=\Delta\big(\mathbf{C}(X_C),\ \mathbf{E}(Y_F)\big)=\Delta\big(\mathbf{C}(X_W),\ \mathbf{E}(Y_W)\big)\,.$$

25. The reader perceives, or after a little practice will perceive, how readily the linkages A' and B alone or suitably combined serve to replace arithmetical by graphical computation of functional expressions,—and it is hoped that he will join with the writer in the following

Concluding Agreements

Today there is general agreement to cut out of arithmetic and algebra many complicated features of limited significance. We make the same agreement as to geometry.

In general, we agree to develop arithmetical, algebraic, geometric technique in a physical and intellectual environment logically and psychologically rich, full of movement, force, color, full of connotations and implications of and for real life of all kinds, including most certainly the real life of mathematics and the sciences.

To secure for our young students a tolerable appreciation of the civilization of the twentieth century on the side of the theoretical and applied mathematical sciences, as teachers of mathematics we agree to shape our instruction in mathematics from the beginning from a point of view no older and no lower than that of the wonderful seventeenth century, and to this end, speaking theologically, we propose to

Canonize the Cross-Section Paper.

THE GRAPH IN HIGH-SCHOOL MATHEMATICS

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This paper is to be regarded as "opening the discussion" of a paper by Professor Moore in this number of the *School Review*.[1] I take my text from the first paragraph of that paper:

(1) Pure mathematics is, as it were, a language for the convenient expression and investigation of relations the most diverse in ordinary life and in nature. (2) The principles of the language are not arbitrary, but are imposed by the *phenomena demanding convenient expression*.[2]

The present paper is confined entirely to secondary mathematics.

In order that mathematics shall be, in the hands of the student, "a language for the convenient expression and investigation, etc.," it is necessary that mathematics shall constantly be used as a language for these purposes, or, more properly, it must be largely *developed* as such language. To speak in particular of algebra: It is certainly true that algebra may be so taught—indeed, in very many instances is so taught—that a formula has little more meaning to the average student than have the mystic symbols of the Masonic Order to the uninitiated. Indeed, there are teachers who stoutly maintain that the best way to teach elementary algebra is to teach the formal operations and formulæ as such, without any serious attempt to give exterior interpretations of them. There are others, however—and the writer is one of these—who believe that the great problem in teaching algebra is to connect its operations and formulæ with life—with the life of the student; to give them meaning in terms of those basic concepts which have already been woven into the learner's mind—concepts which are permanent parts of his intellectual equipment. Two such fundamental concepts are the concepts of space and of number. Professor Moore, in the paper under discussion, is emphasizing the importance of the space concept.

[1] It is taken for granted that those who read this paper have read that of Professor Moore. Hence that paper is not here characterized in full.

[2] The italics and the numbering are mine.

I. NUMBER IN ALGEBRA

I wish first to say a word about the number concept in elementary algebra. One who is not directly engaged in secondary instruction in mathematics may well assume, as a matter of course, that algebra is connected most thoroughly with numbers; its subject-matter is numbers, and no connection is needed. In practice this is not so. Many pupils (probably the majority of them) have no clear idea of the meaning of the equation $ax=b$, and the solution $x=\infty\frac{b}{a}$. I have repeatedly given pupils in the third year of the high school the problem of multiplying 2^7 by 2^9. From 90 to 100 per cent. of the members of these classes get the result 4^{16}. If then they are asked to multiply a^7 by a^9, the answer is unanimously a^{16}. I once remarked on this startling phenomenon to a college professor, who assured me that I must be mistaken as to the facts; while a teacher of mathematics in a prominent high school seriously offered the explanation that pupils have not yet learned to generalize! Probably the most urgent need of secondary mathematics at present is the definite and conscious building of algebra as generalized arithmetic.

A small matter like consistently using the word "number" instead of the word "quantity" would help. The laws of operation in algebra should almost invariably be developed with numbers in the Arabic (Hindu) notation, and then set down in the literal notation to indicate that the *operations* are the same no matter what *numbers* are used.

II. THE GRAPH IN ALGEBRA

By "graph" I mean here any figure usually constructed on ruled (square) paper, which represents phenomena subject to mathematical treatment, and which yields certain conclusions by means of the evident (assumed) properties of the figure. All the properties of the ruled paper are assumed without formal discussion. Intuition and experimentation are given full sway.

Probably one of the most common graphs used in algebra is a rectangle to exhibit the product of two polynomials. To teach the multiplication of polynomials I have found it convenient to proceed as follows: The product $(2+4)(3+5)=6+10+12+20=48$ *because* $6\times8=48$. Illustrate this by a figure. Work a number of problems

of this kind, keeping the problems thus worked out in a collection (on the board) for inspection. By a series of questions the following description of the process can then be elicited: "To obtain the product of two polynomials, multiply every term of one polynomial by every term in the other, and add the products." Then this rule or description of the process is abbreviated into,

$$(a+b+c+\ldots)\,(e+f+g+\ldots)=ae+af+\ldots+be+bf+\ldots\ldots\ldots.$$

In this way the formula comes to represent well-understood facts in arithmetic and in geometry; it is a live formula, not a mystic symbol. With proper guidance and encouragement pupils will now construct figures for the products, $(a+b)^2$, $(a-b)^2$, and $(a+b)\,(a-b)$.

The following may be set down as a general formula for teaching algebraic processes: (1) Perform the operation using plain numbers (Arabic notation). (2) Whenever convenient, construct figures representing the same operation. (3) Write out a description of the process in ordinary English. (4) Abbreviate this description into a *formula*.

But algebraic processes should not be made the chief, nor even the sole, end in teaching elementary algebra. However beautiful and interesting these may appear to the accomplished mathematician, the young pupil finds them of little interest in themselves. The processes acquire interest and importance only when by means of them the pupil is enabled to solve problems which without them would be quite beyond him. What is needed is a set of interesting problems. By using the graph and the intuition freely, we obtain a body of geometric theorems which enable us to apply the most elementary algebra to the solution of highly interesting problems. The theorem of Pythagoras may be proved graphically (the style of proof used applies whenever the two sides of the triangle are commensurable). The altitude, and hence the area, of a triangle can then be obtained when the sides are given. The general case when the lengths of the sides are given in terms of letters is a fine problem for a first-year boy in a high school.

A certain type of good, dutiful student will work faithfully manipulating complicated expressions for the sake of obtaining the answer in the back of the book. A different type of student, usually much abler, will attack a problem like the general solution of the triangle

with all his might, because in solving this problem he obtains the area of *every triangle* at one fell swoop, provided its sides are given. The *problem* is the real thing under consideration, not the process—a problem whose nature is perfectly plain and which is full of interest. The algebraic operations needed for its solution become exceedingly interesting *because* they do things which the pupil wants very much to do, and which he cannot do without them.

The altitude, volume, etc., of a regular tetrahedron whose sides are given can now be obtained (the formula for the volume is, of course, assumed), and furnish the best subject-matter I know in connection with which to study radicals. First use a tetrahedron whose edge is some definite number, say 6; then solve one whose edge is 8, etc.; and then solve the problem in the general case where the length of the edge is a. This gives rise to nearly all the processes in radicals which are taught in the high schools.

In connection with simultaneous equations the graph is of paramount importance. The way to do is not to graph the equation, but to graph the condition which gives rise to the equation. As Professor Moore remarks: "With the cross-section paper in commonplace use, the boy of nine will readily understand and create diagrams of train motions, and will enjoy making the limited express overtake the slow freight at a certain time and place." The boy of thirteen goes farther. He writes a statement which gives the relation between the time since the train started (or some other fixed time) and the distance the train has traveled. He understands that this statement applies to any length of time—that his statement gives a relation between two *variables*, and that this relation is satisfied by an infinite number of pairs of numbers. He then writes an abbreviation of this statement, thus obtaining an *equation* which states the same relation. Thus he has two complete descriptions of the motions of the train, one being the line and the other the corresponding equation. Two lines, representing the motion of two trains, give him the means, as it does the boy of nine, to determine the time and place where the fast train overtakes the slow one; but again the boy of thirteen goes farther. He learns how to manipulate the two descriptions (equations) of the lines so as to obtain from them another solution of his problem. A boy so trained will not speak of *the* value of x and *the*

value of y in an equation $3x+4y=12$. He will not talk nonsense about such things, because to him they are simple common-sense.

In this manner I have had boys and girls in the first year of high school graph the relation between the readings of the different thermometers and obtain the corresponding equations. They understand perfectly well that the graphs and the equations serve to give the reading of one thermometer when that of the other is given. In this, as in most cases, the making of the graph is the simplest part of the work, and should precede the making of the equation.

The above indicates in outline the graphical work that I have undertaken to do with first-year classes.[1]

In a more advanced course in algebra taken in the third or fourth year in the high school the graph is again taken up. It is now possible (if one wishes to do so) to prove that every equation of the type $ax+by+c=0$ is the equation of a straight line (of late I have omitted this proof). The question of the simultaneity of linear equations containing two variables can now be discussed with ease. By analogy we may then also discuss as to simultaneity of the case of three linear equations.

When we come to deal with simultaneous equations of degree higher than the first, the graph is a veritable gold mine. I speak in detail only of the case

$$\begin{cases} ax+by+c=0 \\ x^2+y^2=r^2 \end{cases} \qquad (1)$$

It is very easy for the pupil to convince himself that the equation $x^2+y^2=25$ represents a circle of radius 5 having its center at the origin. The correspondence of the solutions of a pair of simultaneous equations and the intersections of the corresponding graphs are by this time familiar, and we can predict from the graph that, if the algebra tells the whole story, then there must be two distinct pairs of numbers which satisfy both of the equations (1). We are further able to predict the existence of coincident roots; that is, there are lines tangent to the circle, and if we obtain the equation of one such line, and solve it simultaneously with $x^2+y^2=25$, we shall get only

[1] In the mathematical library at the University of Chicago there are notebooks containing the graphical work of first-year classes. These notebooks are faithful copies which pupils have made of their own books.

one solution. It is also easy to see that there is no pair of number which will satisfy the equations $x+y=10$ and $x^2+y^2=25$. Hence the question of imaginary numbers arises in a natural way. (I have thought best not to go into any detailed discussion of the imaginary number in a high-school course. We have simply interpreted the imaginary values obtained when solving the equations as an indication that no real [ordinary] numbers exist which satisfy the conditions imposed.)

The converse problem is of the highest interest, and is of great usefulness in clarifying the meaning of algebraic expressions. Equations (1) are solved simultaneously. The problem then is to substitute for a, b, c, and r such numbers that (*a*) the line and the circle shall meet at two distinct points; (*b*) the line shall be a tangent to the circle; (*c*) the line shall have no point in common with the circle. Predictions are made for certain numbers, and then the lines and the circles are constructed. The straight line is then treated in a similar manner with tbe parabola, the ellipse, and the hyperbola. The solution of a pair of equations like

$$ax+by+c=0$$

$$\frac{x^2}{H^2}+\frac{y^2}{K^2}=1$$

affords excellent practice in solving quadratics, and the pupil sees good reason why he should obtain the solution. The circle and the ellipse treated simultaneously afford a good example where there are four pairs of numbers satisfying the equations.

I summarize the advantages of graphs in algebra as follows:

1. The subject is more interesting throughout.
2. The subject-matter is more clearly understood.
3. The algebra is connected with many other phases of intellectual activity. It becomes "a language, etc." Hence it has an importance to pupils other than as a requirement for graduation.
4. The existence of other branches of mathematics is pointed out, and these are made partly intelligible. Hence it furnishes an incentive to the further study of mathematics.
5. The pupil emerges from the study of algebra with a higher conception of "understanding a subject" than he otherwise would.

Understanding is not so likely in his case to mean "remembering" (as it so often does). It means "insight."

6. No time is lost. It does not require a whit more time to teach simultaneous equations by means of graphs than without them.

The greater interest and the comparative ease with which the subject is grasped more than compensate in point of time for the few recitations required to master the graph.

III. THE GRAPH IN GEOMETRY

In the language of Professor Moore: "Indeed, for the purpose of elementary education our *current deductive geometry is of the nature of a fetish* to be abandoned in favor of a geometry built on a system of axioms rich enough to validate at once the constructional method of the square ruled paper."

The situation seems to be something like this: Pupils who begin the study of deductive geometry have had no experience with logical deduction as such. They have been in the habit of convincing themselves, by means of *all* the resources at their command, that certain statements are or are not valid. To them it is bewildering in the extreme to be asked to "prove" a certain statement by means of certain other statements, and to be careful not to let intuition enter into the proof in the slightest degree. They are required, in the short space of a month or two, to become quite adept, not only in a new subject, but in a new kind of activity—a kind of activity which in itself is very difficult, and which is entirely strange to the pupil. The wonder to me is that they do as well as they do. One does not know whether to admire or pity the gentle docility of a boy of fourteen who struggles gamely to prove that all right angles are equal. He is as sure of the validity of the proposition as he is sure that he is alive. He no more grasps the meaning of "logical deduction from certain assumed axioms" than he did when he was ten years younger.

As a matter of fact, we all fail to carry out the avowed Euclidian program in our practice. We do better than we pretend. Our proofs, especially in the first part of the course, are by no means strictly deductive. Indeed, they are much farther from being so than is currently supposed.

It is proposed, as I understand it, that, instead of carrying out the

Euclidian program from the beginning, we shall consciously adopt a quite different course. My own practice has been as follows:

a) Write down a list of propositions about space (in terms of points and lines in a plane). This list includes all the statements we can find about whose validity we are all agreed there can be no doubt. Every statement about which there is a doubt is ruled out. The certainty as to their validity must arise from what we all see *has to be so*, and not from some extraneous authority. These propositions should be carefully worded and numbered. (We refer to this list later on as list (A).)

b) In this process we soon come to propositions of which we are not certain. (Most of the propositions which we discuss are found in our text as theorems, corollaries or exercises.) Then we may satisfy ourselves of their validity by careful construction. Since we have the square paper, all its obvious properties being included in list (A), we can now begin to mention that the properties in question are *exactly* true, if the propositions of (A) are exactly true, much more exactly true than we can ever verify by actual construction. "It must be so," says the pupil. That is his first "proof;" but little does he recognize it as that suggested or written out in full in the text. In this manner we proceed learning geometry. Intuition, construction, argumentation, all have their natural sway. But argument (proof) becomes increasingly important as we go on. A list is kept of these propositions which are established by construction. (We refer to this list here as list (B).) It is surprising how soon list (B) is completed. We get so that we can *see* whether propositions are true or not without constructions, i. e., we are *proving* them. List (A), however, continues to receive additions throughout the year.

Great freedom is given as to the means of proof. If an algebraic proof is more convenient than a so-called geometric proof, then use algebra. If the use of algebra involves new propositions, we only need to make provisional or permanent additions to list (A). *Let the pupil have his way* about the kind of proof he is to use. The great thing is to find a pupil who has a *way* at all. The less hopeful ones are those who are always waiting for the teacher's way. Never stop one pupil who is making a proof because another pupil insists that he has a shorter and more elegant one. If possible, get several

proofs. The pupil who has worked out a difficult one will understand a simpler one easily enough.

When the number of pages allotted for the year's work have been covered in this manner, the pupil has *lived* himself into somewhat of an understanding of what is meant by logical deduction. If thought desirable, the question can now be raised as to whether some of the propositions of the lists (A) and (B) cannot be logically deduced from the remaining ones.[1] If this is not done, there will be considerable time left for working out special problems, every one of which requires deduction from theorems already proved.

I sum up the advantages of such a course as follows:

1. The pupil is not required to spend a good share of his time trying to do what he cannot possibly do.

2. He feels vastly more cheerful, which is a great desideratum both from the point of view of good health and from the point of view of possible progress. He is not quite so ready to shake the dust of mathematics off his feet.

3. He gets acquainted with a larger body of mathematics, and sees more of its relations to other subjects, because he has more time.

4. He is aware of the existence, among other problems, of a large problem in logical argumentation, viz., the reduction of the lists (A) and (B) to the smallest possible proportions. He has not finished the subject.

IV. THE NEGATIVE NUMBER IN ALGEBRA

"The principles of the language are not arbitrary, but are imposed by the phenomena demanding convenient expression." (See p. 317.) Under this, the second part of our text, I wish merely to say a word about the negative number. In trying to define algebra so as to distinguish it from elementary arithmetic, it is often said that the presence of the negative number is the distinguishing characteristic of algebra. This may be true formally, but in reality the negative number is present in arithmetic. Instead of speaking of positive and negative latitude, we speak of latitude north and south of the equator. In finding the difference between two latitudes, one north and the

[1] In the early part of the year this would have been a useless question, since the pupils did not know how to deduce. And they could no more learn by being told than they could learn to swim or ride the bicycle that way.

other south, we do exactly the same thing that we do when we find the difference between a positive and a negative number. It so happens that in dealing with many different phonomena in our environment it is convenient to start at a certain point, not an end point, and measure, so to speak, in two directions. It is for the "convenient expression and investigation" of such phenomena that the negative number is introduced in elementary algebra. The laws of operation with negative numbers are best obtained, for the purposes of instruction, solely by induction from concrete examples to which the negative number is applied. When we once admit that the principles of mathematics are imposed by the "phenomena demanding convenient expression," this mode of treating the negative number is obvious. I give just one example to illustrate the mode of procedure.

$$500 \text{ gain} + 300 \text{ loss} = 200 \text{ gain}$$

When *translated* into the language of algebra (under the convention that a positive number shall represent gain and a negative number shall represent loss), we have

$$+500 + (-300) = +200$$

CONCLUSION

As in geometry, the widespread belief that our current texts are rigorously and infallibly logical throughout has, in my judgment, done much to retard the progress of teaching; so in algebra, the feeling that the laws of operation derive their sanction from on high rather than from man himself, and through him from his environment, has done much to keep us from taking a common-sense view of the situation. As in many other cases, so here, the simplest point of view is also the most profound.

I have commented on Professor Moore's paper so far as I have had experience with the things he proposes. Everything mentioned in this paper has been tried repeatedly in classes. I have tried a good many other tricks with graphs, etc., which at the time I thought were fine; but as the deepest-laid plots often fail, so our finest appearing plans will at times go wrong. This has repeatedly been my experience. I have been able to make progress but slowly. In the course of another year I may be able to fit into some course the

parabola $y=x^2$ for the purposes of construction somewhat as suggested by Professor Moore. It is always my feeling that we are most truly reformers—we are most truly followers of the champions of reform—when we become imbued with the feeling that things may and will be made better than they are at present. It even sometimes happens that, to be the followers in the highest sense of those who are pointing the way to better things, we are compelled to do certain specific things which are diametrically opposed to those they advocate. Every individual teacher must become a center of reform, in the main working out his own salvation. Suggestions and stimulus from others are of course indispensable, but the suggestions can seldom be worked out as they were intended by him who gave them.

I wish to say, finally, that as a result of six years' experience in teaching elementary mathematics in a high school, I am firmly convinced that graphical work is of very great importance in creating interest, and in promoting a clearer and, to the student, a more satisfying insight into subjects which but too often are mysterious riddles.

SPELLING AND ARITHMETIC IN 1846 AND TODAY

V. W. B. HEDGEPETH
Superintendent of Schools, Goshen, Ind.

Because of a very opportune find by Principal Riley, of Springfield, Mass., the most satisfactory comparison of the school methods of our grandfathers' time and of our own was recently made possible. While rummaging in the garret of an old school building, complete sets of examination papers, together with the answers and markings, all bearing the date of October, 1846, were found, and are today in a bound volume in the office of the superintendent.

By submitting these questions to pupils of the same age, and comparing results, the school authorities in Springfield were enabled to arrive at some rational comparison as to the progress or retrogression of the school system in Springfield at least.

Originally the questions were given to pupils of the ninth grade, which would correspond to the freshmen class of our own high school. In order to make the test absolutely fair, they should have been submitted to the high school rather than to the eighth grade.

The writer often has been called upon to dispose of the charge that in general our grandfathers as children were better spellers and better arithmeticians than the children of our own generation. At their best, heretofore the champions of the old and the new have not been able to find any fixed basis for sound argument, but have proceeded from their own conclusions, which are often colored by personal prejudice.

On December 20 the questions were taken verbatim from the list of 1846 and submitted to the pupils of our eighth grade, both eighth B's and eighth A's. The time consumed in the arithmetic examination was approximately forty minutes, and in spelling about twenty minutes. The questions were given without any preparation whatever on the part of the children, without any previous notice to teacher or pupil, and without any intimation as to why the exam-

ination was called, or any information as to the source of the questions. The following were the questions in arithmetic:

1. Add together the following numbers: Three thousand and nine, twenty-nine, one, three hundred and one, sixty-one, sixteen, seven hundred two, nine thousand, nineteen and a half, one and a half.

2. Multiply 10008 by 8009.

3. In a town five miles wide and six miles long, how many acres?

4. How many steps of two and half feet each will a person take in walking one mile?

5. What is one third of 175½?

6. A boy bought three dozen of oranges for 37½ cents, and sold them for 1½ cents apiece; what would he have gained if he had sold them for 2½ cents apiece?

7. There is a certain number, one third of which exceeds one fourth of it by two; what is the number?

8. What is the simple interest of $1200 for 12 years, 11 months, and 29 days at 6%?

In 1846 the average percentage of correct answers was 29.4. In Springfield in 1905 the average percentage of correct answers in this same examination was 65.5, and in Goshen the average of correct answers reached the remarkably high grade of 87.8 per cent.

Following is the list of words to be spelled:

accidental	deceitfully	feignedly	imbecility
accessible	descendant	ghastliness	inconceivable
baptism	eccentric	gnawed	inconvenience
chirography	evanescent	heiress	inefficient
characteristic	fierceness	hysterics	irresistible

It will be noticed that their array is rather imposing and would be formidable even to the reader.

The average percentage of correct answers in both subjects was: 1846, Springfield, arithmetic, 29.4; spelling, 40.6; 1905, Springfield, arithmetic, 65.5; spelling, 51.2; 1905, Goshen, arithmetic, 87.8; spelling, 46.2.

On careful examination of the existing conditions at the time of these tests, the results lead us to several conclusions.

In 1846 the Springfield school year consisted of 44 weeks of actual school work, each school day containing 6 hours. In 1905 in Springfield the year consisted of 40 weeks of 5 hours each day. In other words, in 1846 the schools were in session about 1,340 hours; in 1905

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4. How many steps of two and half feet each will a person take in walking one mile?

5. What is one third of $175\frac{1}{2}$?

6. A boy bought three dozen of oranges for $37\frac{1}{2}$ cents, and sold them for $1\frac{1}{2}$ cents apiece; what would he have gained if he had sold them for $2\frac{1}{2}$ cents apiece?

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the schools were in session about 1,000 hours. In 1846 the schools were among the best, as they had been the first to have a regularly appointed superintendent; they were entirely without any foreign, non-English-speaking, element, had been highly complimented by Horace Mann, and were under excellent supervision. Their course of study was quite definite, and consisted daily of reading, writing, arithmetic, geography, and spelling. Spelling especially was strongly emphasized, as the following extracts from the course of study at that time show:

> Accuracy in spelling and excellence in reading are deemed of the first importance.
>
> Ability to spell correctly is deemed highly important, as lying at the foundation of all requirements, without which no person can be accurate or intelligible as a scholar, or ever safe from exposure to great mortification in after-life.

At the present time in Goshen the school year consists of 36 weeks of 5 hours per day, making the year contain 900 hours. In both cities the children are supposed to enter the grades at 6 years and high school at 14 years of age. In reality the children who took the examination in 1846 had been in school correspondingly 3 years longer than their Springfield grandchildren in 1905, and over 4 years longer than the Goshen children who last month wrote this examination. Nevertheless the pupils of 1905 reached a much higher percentage of effectiveness than their grandfathers of 1846. This is shown, not only in the total of correct answers, but in reduced differences. For example, in 1846:

> More than one-fourth of the examples were passed over as too difficult to attack, and the incorrect answers were so far from the mark as to overwhelm one with the conviction that the children were entirely lacking in power to mentally approximate the results. Answers to the fifth example varied from $5\frac{1}{2}$ to 6,312. Below are some of the incorrect answers to the problem in simple interest—a problem which was worked correctly by only thirteen pupils. Dollar signs, decimal points, and commas are the pupils', the first two conspicuous chiefly by their absence:
>
> $87.58.00; $93.58; $114.00; $179.80; 907.92; $937.80; $9328.00; 93.28; 96.86; 115.08; 2.15.80; 449.50000; 475.00; 638.00; 932.200; 1860,58; 93,580; 491040; 892800; 31966 $\frac{2}{3}$; 1908000; 110, 88, 05; 4593600; 5587200; 77001740o; 11038980000; $72\frac{1}{2}$.
>
> Less than one-half of the class got the correct answer to the first example; fifty had the second correct; only eleven secured the desired result in the fourth; and 7—all boys—obtained the mastery in the fifth. Of 39 girls, not one had

the right answer to the fourth or sixth, and only 3 worked the interest problem to a successful conclusion. The girls averaged 9 per cent. on the test.

In Goshen only 5 pupils missed solving the first example correctly. The fifth was solved by all but two—both boys—who gave as their answers 351 and 526½ respectively. These appear rather large numbers to be one-third of 175½. With reference to the interest problem, 8 pupils omitted it altogether, and 7 solved it incorrectly, all the rest of the class having presented a correct solution. The 7 incorrect solutions were: $25,991.80; $3,218.00; $896.80; $887.80; $467.90; $978.00; $869.80.

Out of a total of 46 who made 100 in arithmetic, 26 were boys and 20 were girls. The whole class of 101 was composed of 54 girls and 47 boys. These comparative results in arithmetic speak for themselves.

Equally interesting are the results in spelling. For instance, on the old papers the words "heiress" and "baptism" were spelled in the following ways:

heirress	airress	areress	baptisim
hurriss	airess	arerest	baptisimn
heirruss	airest	eiress	baptisem
heirees	airresst		baptisom
heirness	airhess	babtism	baptisum
hieress	arress	babtisism	baptisemn
heress	arris	batism	baptysm
hirress	aries	batisim	baptisiam
hereis	ariest	baptsim	baptiasm

Our own papers show some brilliant flashes of originality, but with some differences. In the old papers we find that 31 pupils out of 85 misspelled "baptism" in 15 different ways. The word "heiress" was written by 43 pupils in 22 different ways. In our own papers we find that 31 pupils out of 101 mispelled "baptism" in 20 different ways, and nine pupils misspelled "heiress" each in his own way. The list follows:

babtizm	baptizam	babtazim	hierest
babtizimn	bathizthm	babtisum	harriss
babtismn	batistion	babtisem	earess
babtisiom	babptisim	babystism	hirress
batisim	baptysm		herris
baptisim	babtism	hirest	airasess
baptisism	baptizem	eiress	heirous
bapitisan			

Notwithstanding the fearful and wonderful ways in which our children managed to misspell "baptism," we have the satisfaction of knowing that, with approximately 4 years less schooling they raised the total percentage of correct answers from 40.6 to 46.2.

When we consider that originally the questions were set for pupils who heard English in all their homes, who attended school one-third longer each year than now, who had fewer studies and hence more time for each, who had been during the five preceding years under able principals, were in a school in which the younger and backward pupils had been taken out nearly two years before, and were of an average age with our present high-school freshmen, the results of the examinations in 1905 present to the thoughtful critic a number of very plain facts. The results establish the superiority of the modern schools in spelling and arithmetic. If the test had been set for pupils in Springfield alone, this conclusion might be questioned, but it would be difficult to prove the incorrectness of it, since the pupils of two representative schools so far apart as Springfield and Goshen show uniformly greater grasp and increased power.

Nor is it difficult to explain the reasons for the superiority of the modern school. In 1846 11 different books, excluding geography, history, and physiology, were read below the high school. Today our children read from 70 to 80 in covering the same course. Our school libraries are filled with books that are not only informational, but have their substance arranged in a manner to stimulate and arouse the interest. These books cover a vast range of subjects, and in their reading the child acquires unconsciously a large vocabulary, an increase of power over the spelling of words, larger opportunities, is broadened, enriched, increased in capacity for enjoyment, and widened in his sphere of usefulness.

There can be little question that the school today is vastly superior in efficiency to the school of our grandfathers' time; and in my own mind there is equally little question as to the fact that in like manner, following the incorporation of the gymnasium, the bath, domestic science, and manual training, will the schools of our grandchildren be superior to our own.

A MODERN ROMAN STATE

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In the preceding paper I described in detail the organization of the Roman state and the steps necessary in its formation. In the following I shall attempt to show how this organization, in itself but a skeleton, became a living organism through the interest excited by the campaign, the election, and the subsequent activities of the state and its officials. This division, however, is logical, not temporal; for each step in the former process was associated with a corresponding development in the second.

In the progress of the campaign several stages may be noted. First comes a period of desultory canvassing (*ambitus*) and booming of candidates, corresponding to the period before our party conventions, and beginning an indefinite period before the election (e. g., Cicero began his canvass nearly two years in advance). This stage is terminated by the public declaration (*professio*) on the part of candidates of their intention to run for office. While in Pompeii *professiones* could be made up to the day before the election, it is probable that at Rome they were required in time to be included in the edict of convocation, which thus terminated the preliminary campaign, and corresponds roughly to our party conventions. Then follows the real campaign, which is closed by the election itself. I shall describe in succession: (1) the preliminary campaign; (2) the *professio;* (3) the edict of convocation; (4) the election; (5) the entrance into office.

The list of officials elected by our state has grown during the three years of its existence, till this year we elected a complete set of civil, military, and religious officials, as follows: civil: 2 *censores*, 2 *consules*, 8 *praetores*, 10 *tribuni plebis*, 2 *aediles curules*, 2 *aediles plebis*, 20 *quaestores*, 26 *viri;* military: 2 *viri navales*, 24 *tribuni militum;* religious: 15 *pontifices*, 15 *augures*, 15 *viri sacris faciundis*, 7 *viri epulones.*

There are thus 72 civil officials, 26 military, and 52 religious, making a total of 150. The first step was to instruct the citizens as to what offices they were variously eligible. We are gradually establishing the ancient *cursus honorum*, and this last year no one ran for consul, unless he had previously been elected prætor, or had been defeated for the consulship. The ædileship or quæstorship was made necessary a preliminary to the prætorship. Next year the complete *cursus* will be put in force, consisting of quæstor, prætor, and consul, with the holding of a subordinate office, civil or military (*viginti-sex vir, tribunus militum, vir navalis*), a desirable, although not indispensable, qualification. These regulations produce, for the civil and military officials, the following general scheme: First-year pupils are eligible to the offices of *viginti-sex vir, tribunus militum, vir navalis*, and occasionally quæstor; second-year pupils, to quæstor, ædile, *tribunus plebis*, and prætor; third-year pupils, to prætor, ædile, *tribunus plebis*, consul, and censor; fourth-year pupils, to consul, censor, and *tribunus plebis*. The ancient provision that one censor and one consul should be plebeians, and that both might be, is imitated by the regulation that one consul and one censor must be third-year pupils, and that both may be. The practical object of this is to insure the presence here next year of at least one of the consuls and one of the censors elected this year.

The religious officials, as respects tenure of office, have not yet been placed on a basis strictly Roman. Thus far they have been elected each year, but as soon as every department of the state is working smoothly, we shall probably substitute life-tenure; i. e., a successful candidate will hold the office so long as he remains in school. Annual re-election at first serves, moreover, to secure greater familiarity with these offices than would the immediate introduction of the actual conditions. In general, first-year pupils may become *VII viri epulones* or *XV viri sacris faciundis;* second-year pupils, *XV viri sacris faciundis* or *augures;* third- and fourth-year pupils, *augures* or *pontifices*. The *pontifex maximus* is always the fourth-year pupil that carried the largest number of tribes. When life-tenure is introduced, the *pontifex maximus* will thereafter be elected by the *pontifices* themselves from their own number.

Each class was first informed of the offices to which its members

were in general eligible, and the functions of these officials were briefly described. We limited our efforts in each class to just those officials with whom it was concerned, and thus, in the beginning classes, but five or six were discussed. The *principes* in particular were carefully instructed as to the eligibility of their *magistri* and *discentes;* and they, especially in the lower classes, guided the choice. Each *princeps* was furnished a list of the officers concerned, a printed description of them, and the approximate number that should run in his guild for each office — a number chosen to secure, in general, three candidates for every office. Then began a period of preliminary discussion and canvassing among the pupils, the result of which was to clarify their ideas as to the relative importance and desirability of the various offices, and thus to determine their choice. This period was marked by the appearance of the first inscriptions, which from that point form an essential element in the campaign. For these inscriptions the election notices found at Pompeii are made the basis. The first-year pupils are given some account of Pompeii, of its destruction and excavation. At this point, too, they are introduced to Bulwer's *Last Days of Pompeii*, the reading of which is required during the first year. They are then made familiar with typical forms of inscriptions, which indicate clearly that in theory, however it may have been in reality, the office sought the man. The candidate never takes the initiative, never announces his own candidacy, never speaks in his own name. It is his neighbors, friends, and relatives who assemble, discuss and weigh his merits, his chances of success, and decide that he shall run for office. Not until their opinions have been widely expressed through the only newspapers they possessed, the inscriptions, does the candidate formally announce his intention. These inscriptions are found all over Pompeii, painted on the walls of houses and shops, usually in red, about four feet from the ground, in letters of varying regularity. Owners of buildings derived considerable profit, evidently, by renting space to candidates and their friends. Inscriptions belonging to the preliminary campaign purport to have been written by neighbors, friends, or clients of a Pompeian citizen, urging him to run for this or that office, and contain four essential elements: (1) name, in accusative case, of the person who is urged to run;

(2) name, in accusative case, of the office; (3) name, in nominative case, of person or *collegium* urging him to run; (4) verbal phrase. E. g.: *Casellium Marcellum aed(ilem) rog(ant) vicini*—the neighbors of Marcellus urge him to run for ædile; *Trebium aed. tonsores (rogant)*—the guild of barbers urges Trebius to run for ædile; *Celsum aed. Genialis rog.*—Genialis urges Celsus to run for ædile. The verbal phrase is usually a simple rogant, abbreviated *Rog. R.*

We found it necessary to prescribe that the English name of the person proposed should be used with Latin terminations, and the Latin name of the one proposing him; for the former must be intelligible to all who read the inscriptions, if his candidacy is to be promoted, while the latter is of secondary importance, and of value mainly in familiarizing each citizen with his own name. At this time the ædile found it desirable, as general superintendent of public streets, buildings, etc., to issue a proclamation—in Latin, of course—defining the hours within which inscriptions could be posted, and under what circumstances they could be erased, limiting strictly the available spaces, and forbidding at the same time any excessive demonstrations. He also attached a heavy penalty to the introduction of any modern and foreign (i. e., English) barbarisms into the purity of the Latin tongue.

Meanwhile each citizen is deciding for what office he shall run, and is assisted materially in his choice by the action of the political parties. The development of the political parties in our state has been very interesting, for it has been entirely spontaneous, and thus indicative of genuine political spirit. The first year the nearest approach to a party was the mutual support given one another by about ten candidates, who did some effective electioneering, and on election morning circulated a ticket comprising their names. Every member of this ticket was elected, and its success suggested to others the formation of similar combinations the following year. One of these combinations began heading all their notices with the legend *S. P. Q. R.* and, since this suggested conservatism, at length called themselves *Optimates*. Thereupon their rivals assumed ancient names till these were exhausted, and then invented new ones. There were five in all: *Optimates*, *Populares*, *Equestres*, *Res Publica*, and *Combinantes* (*sic*). This year we succeeded in limiting them to three

ancient parties: *Optimates*, *Populares*, and *Equestres*. As a result, there are now established fairly well-defined party traditions and a recognized party leadership. The rivalry and competition among them, in their desire to pledge the best candidates to their party, was itself sufficient to orient the entering pupils and initiate them into the campaign, especially since each party drew up a complete set of candidates for each office. In this respect we deviated slightly from ancient conditions, and in place of real principles to bind together the members of a party, we were obliged to substitute a more definite organization than probably existed in the days of Cæsar and Cicero. This procedure was fairer to all concerned, since it insured, for example, sixty candidates for the quæstorship, all starting on an equal basis; whereas, if each party supported but a limited number, six or seven, the twenty on the three tickets would not be opponents at all, but would doubtless all be elected to the exclusion of the forty not on any ticket. This method, moreover, secured the three candidates desired for each office and insured the active interest of 450 candidates for the 150 offices. At the same time, it made the propagation of this interest through the remainder of the school an inevitable result, since each party, starting thus with 150 adherents, must depend for victory upon winning supporters among the 400 citizens not running for office. It follows that no one would care to run unless he was on one of the three tickets; but this was practically no injustice, for each party had difficulty in filling the last half-dozen places. When the party tickets were complete, and it was evident that everyone had reached a decision, the candidates made their *professio*, or formal declaration to the official that was to preside at the election, of their intention to run for a stated office. To avoid confusion, however, the *consul*, *praetor*, *tribunus plebis*, and *pontifex maximus* sent representatives to each class, and the candidates rising gave their names and the offices involved to these representatives, who passed upon the eligibility of each candidate, and finally published a complete list of the candidates for each office.

This preliminary campaign is brought to a close by a proclamation of the *consul* at least seventeen days before the date selected for the election, summoning the *comitia centuriata* to meet in the Campus

Martius, and enumerating the officials to be elected by this assembly, viz.: 2 *censores*, 2 *consules*, 8 *praetores*.

The 8 *praetores* have meanwhile met and appointed one of their number, either by *comparatio* (agreement) or *sortitio* (lot), to issue a corresponding edict, summoning the *comitia tributa* and enumerating the officials to be elected by that assembly, viz.: 2 *aediles curules*, 20 *quaestores*, 26 *viri*, 2 *viri navales*, 24 *tribuni militum*.

The *pontifex maximus* then issued a similar edict, stating that on a certain day he would choose by lot 17 of the 35 tribes of the *comitia tributa*, and that these would proceed to elect the religious officials, viz.: 15 *pontifices*, 15 *augures*, 15 *viri sacris faciundis*, 7 *epulones*.

A *tribunus plebis*, selected as was the *praetor*, issued the proclamation summoning the *concilium plebis* and enumerating the offices to be filled, viz.: 10 *tribuni plebis*, 2 *aediles plebis*.

The same day was appointed for the meeting of all the electoral assemblies and the election of all officials—an abbreviation obviously necessary, although technically contrary to the ancient custom.

During the period following the *professio* and the edict of convocation, the campaign waxed hotter and hotter. The characteristic forms assumed by the *petitio*—i. e., the *prensatio*, *salutatio*, etc.—were to some extent imitated this year, and at the next campaign we expect to see the *candidati*, dressed in their togas and attended by their *nomenclatores*, *clientes*, etc. The greatest rivalry centered about the *principes*, for the party leaders felt that in gaining a *princeps* they had captured the most influential member of his *collegium*, and one that could probably carry it for them. The *princeps* consulted with his *magistri*, the *magistri* with the *discentes;* and the fruit of it all was a clear understanding of the ancient party system. The most conspicuous product of this period were the inscriptions, the fourth element of which now changes from a verbal phrase asking someone to run for office, to a phrase urging the people to vote for him. The letters *O. V. F.* constantly appear instead of *R.*, and represent *oro ut* (or *vos*) *faciatis*. Furthermore, the simple forms, employed mostly by first-year pupils and erased from the boards from day to day, develop through all possible intermediate stages into elaborate inscriptions in permanent form. Individual

candidates are extolled, promises made, opposing parties attacked, the claims now of one party, now another, presented to the citizens. Some really artistic work in painted inscriptions was done, while the amount and variety of the Latin employed would rival some of our prose books. A favorite mode of advertising was to parody well-known passages of Cæsar, Cicero, and Vergil. The first chapter of Cæsar especially lends itself to such treatment with the three parts of Gaul corresponding to the three parties, all of whom adapted it to their use. Here follows a portion of a *Popularis* adaptation:

Schola est omnis divisa in partes tres; quarum una Populares, alia Equestres, tertia Optimates appellatur. Hae omnes sensu, animo, opinione inter se differunt. Harum omnium honestissimi sunt Populares, propterea quod a facinore et stultitia Equestrium et Optimatium longissime absunt. Populares credunt et pueros et puellas in gubernaculo aequam partem habere debere. Itaque omnes boni cives, et pueri et puellae, ferte suffragia ad Populares et create Lowenthalum et Coddingtonam consules.

The value of these inscriptions cannot well be overestimated. To express a natural thought arising from one's personal experience, to further a real purpose, by the composition of a Latin sentence, is to most of our high-school pupils a rare experience and, when once felt, gives them a new feeling for the language. The question of Latin prose is always with us. No one doubts that the expression of natural ideas, ideas within the experience of the pupil, would be an ideal solution, did it not at once require a vocabulary, if not absolutely impossible, at least totally inconsistent with the requirements of the texts read. But in our state and its election we have a subject-matter that is within the immediate range of the pupil's experience and of interest to him, while the vocabulary required is strictly Cæsarian and Ciceronian. Time will be necessary to test the value of such a solution, but the experiment has opened up interesting and suggestive possibilities along that line. Imagine, for example, a state trial, such as we expect shortly to have, involving a case familiar to the pupils, acted out by them under the presidency of a prætor, and including some exhibition of the tribune's veto power, or the pardoning privileges of the Vestals; imagine the pupils passing to their classes and being expected to describe in Latin the scene which they witnessed, and in which they participated.

You will need scarcely an expression not already found in Cæsar and Cicero, while all that is most valuable in the objective method of teaching modern languages will be incorporated.

For the actual conduct of the campaign each party formed a committee corresponding to the ancient *divisores*, who played a very important, and not necessarily illegitimate, part in the ancient elections. Each committee was composed of 29 *divisores*, one from each *collegium*, who looked after the interests of his party there. Various modes of advertising were invented. Borrowing from another field of Roman life, the various parties chose colors—the *Optimates* purple, *Equestres* green, *Populares* yellow; and soon these colors were everywhere. It was incidentally a proof of the interest taken by the pupils, that in the very midst of the football season, in a school famous for its football enthusiasm, with a team not yet beaten, the display of party colors for the three weeks preceding the election quite eclipsed those of the school. One afternoon all the parties held mass-meetings simultaneously, and over half of the pupils were in attendance at one or another. That anything in their study of Latin should so arouse their interest that such a proportion should voluntarily remain after school hours to further its success, is still another indication that the political atmosphere of ancient Rome is the one most adapted to bring the pupil into sympathy with Roman life. Numerous other devices were resorted to by the various parties in furthering their cause. One secured arm-bands with the party colors and initials; another decorated the entire assembly hall with purple bunting; while the *Populares*, true to their character, fed the populace at lunch period with 2,000 cakes. The campaign fund of each party was limited to $25, collected by voluntary subscriptions from its 150 candidates.

It was our constant aim to concentrate the interest of party rivalry upon each essential element in succession, a process the inevitable result of which was a clear grasp of every step in the organization, so closely is the proper manipulation of the campaign dependent on such an understanding. Thus, after the contest involved in drawing up the tickets had insured accurate knowledge of the names and grades of officers, the center of the conflict was transferred to the tribal organization, and to the election by each

tribe of a *curator tribus* and *tribunus aerarius*, the former of whom takes charge of the tribe at elections and the latter of its finances. Their selection at a preliminary election was naturally regarded as a test of party strength and as foreshadowing the final result. Consequently all the energy of the party managers was devoted for several days before this struggle to educating their followers, instructing them as to their duties — a procedure that at once secured a thorough comprehension of the tribal organization. This election took place one week before the final contest and occupied fifteen minutes, with the assembly organized as a *comitia tributa*. The prætor who presided had appointed as assistants 35 other officials — prætors, quæstors, etc. — each of whom conducted the election in a tribe. The 35 *curatores* elected were then summoned and instructed carefully as to their duties at the election.

Shortly before the election each party was allowed to send a representative to each class to present its case and seek support. He was introduced by the *princeps* and allowed just three minutes. The speeches, which were naturally delivered by the leading candidates on the various tickets, were partly in English, partly in Latin, and almost every one of the fifteen speakers showed a correct conception of his party's position in the state, and considerable familiarity with its history and leaders. On the next day each *collegium* held an election, at which it selected one candidate for each office (i. e., one for consul, one for prætor, etc.). Of these candidates the *collegium* became the *patronus*, and all the members were expected to support them on the final ballot. This reduced slightly the individual's freedom of choice, since 14 of the 150 candidates were prescribed by his *collegium*. The result of this election was practically to settle the choice of one consul and one censor. The parties made careful plans for this vital contest, and the various representatives in each guild endeavored to induce all members of their party to agree upon candidates and thus carry the guild.

Meantime the parties had been preparing their tickets and platforms, and during the week previous to the election these were issued in printed form in their proper colors. It is interesting to note that, without any suggestion, the platforms assumed a characteristic form. That of the *Populares* is of extreme simplicity, to be within reach

of the masses; while that of the *Equestres* is much more elaborate and pretentious — an adaptation of Cicero. They were composed by fourth-year pupils, and stand as they were written. Following are the platforms ef the *Populares* and *Equestres* (the *Optimates* printed merely their ticket without a platform):

Nos Equestres haec, Quirites vobis pollicemur !

Primum — Summam in consulibus fore diligentiam summam in senatu auctoritatem, summam in equitibus Romanis virtutem, summam in omnibus bonis consensionem.

Deinde — Consules nec tempus ad festos dies celebrandos, nec tempus ad luxuriam nec pilam et alias voluptates, nec denique ad quietem animi et corporis sument sed videbunt ne quid civitas detrimenti capiat, et omnes cives diligenter ad summam rem publicam se incumbant.

Maxime — Consulibus Boydo et McMatho creandis omnia vetera mala rei publicae oppressa et vindicata esse et secundas res et pacem appropinquare templum Jani intercludi et aedificia pulcherrima in Forum ventura et doctrinam et litteras vigentes videbitis.

Denique — Atque omnia haec sic adminitrata erunt ut Jovis Optimi Maximi nutu gesta esse visura sint et consules sicut aliquos non ex hac urbe dilectos sed de caelo delapsos; et vos Quirites semper vivere velitis.

PARS POPULARIS

Haec est pars una in re publica quae est populi et populo; pars sola quae aequam iustitiam omnibus det. Aequitati et iustitiae omnibus temporibus stetit. Huius partis fuerunt multi clarissimi viri, quo numero maximi fuerunt Gracchi et Drusus et Marius. Hanc partem Caius Julius Caesar ad victoriam duxit.

Si candidati popularum creabuntur, tota res publica pace et serenitate fruetur, et omnes cives beati erunt.

Nostri consules et censores nulla mala patientur, et potestatem Romae extendent.

Si summum bonum Romae vultis, suffragia ad hanc partem fertote.

Since the preparation of 150 ballots would, if deferred till the actual hour of election, require too much time, they were distributed the day before and filled out in advance. We have deviated in several points from the ancient mode of balloting. In the first place, the election was concluded in one day, contrary to the actual custom, according to which the elections must sometimes have occupied weeks; secondly, to avoid the necessity of second elections, a plurality only was required, and the two candidates for censor receiving the highest number of votes were, for example, declared

elected; furthermore, in the *comitia centuriata* the vote of centuries consisting of but two citizens would, whenever these two disagreed, have to be decided by lot, and as this would require greater accuracy than we could hope for from 373 centuries, we substituted for separate ballots, counted and reported by centuries, a general centuriate ballot, containing spaces for two *censores*, two *consules*, eight *praetores*, inscribed by each voter with the name of his class, tribe, and century, to be collected by classes and counted as described below. Another year we shall be able to follow the ancient form more closely. Similar ballots, perforated to assist in counting, were prepared for the *comitia tributa*, the modified form electing the religious officials and the *concilium plebis*. Since these last were to be collected and counted by tribes, they needed to give no further information than the names of the candidates.

The only occasion on which our Roman state interfered with the regular school session was election day itself, when three of the six periods were omitted. The question, however, will naturally arise as to how much class time has been required for the explanation and organization of the state, and for the campaign preceding the election. It should, in the first place, be borne in mind that the form of our state here described is the result of three years' growth, and that each year the pupils have needed instruction only in the additional developments, not in the entire organization. If transplanted, it should be by a process resembling its original growth. In the second place, the idea at the basis of the experiment anticipated — and on good grounds, as results proved — the spontaneous generation of an interest thriving better outside the classroom than in it, that would make the essential elements as familiar to the pupils, and from similar causes, as the schedule and the stars of his football team. Even if the most important period in the life of the state, which will always center about the campaign and election, did for a month cause a marked loss in the amount of ground covered in the textbook, I should maintain that the exchange were a profitable one, and that, furthermore, the amount of text read, if that be made a criterion of progress, would ultimately be greatly increased by the greater interest, and consequently more rapid advance. As a matter of fact, the work of my own classes was not interrupted at all, while

the other Latin instructors estimated the cost at from one to three chapters of Cæsar, one to two chapters of Cicero, and two lessons in the beginning classes.

Two or three individual pupils were found who had permitted their interest to interfere with their other work, but, aside from the ease with which any such cases can be checked up, it was to me very refreshing that, amid the multitudinous and wholly extraneous interest by which pupils are today distracted, one could arise with equal spontaneity, awakening and absorbing their interest to the same degree, while at the same time forming an integral part of their classical studies.

In the conduct of the election this year we attempted for the first time to observe, so far as practical, the due formalities. Early in the morning a large red flag was hoisted beneath the Stars and Stripes, representing the flag that, hoisted on the Janiculum, originally assured the army, assembled as a *comitia centuriata* in the Campus Martius, that the fort was properly guarded. Later it became merely a signal for the people to assemble. Then the trumpet was blown at the four corners of the building, which thus represented now the city walls, now the *arx*, and finally from the rostrum where the consul was to preside. These formalities were essential to the convocation of the *comitia centuriata*. The signal bells were dispensed with, and at 9 A. M. heralds (*praecones*) proceeded from room to room, summoning all citizens and non-citizens alike to a *contio* in the Campus Martius. This preliminary *contio*, which, as I assumed — although the question is undecided — always preceded an electoral assembly, was of course unorganized. After a few introductory and explanatory remarks, the consul entered, preceded by the twelve lictors with *fasces*. We have established the tradition that these lictors should include all the candidates for the consulship. The chief lictor cried out the approach of the consul, and a way was made to the platform, where the lictors lowered their *fasces* in the presence of the sovereign people. The consul at once took charge, and, to secure an opportunity of introducing the *auspices*, assumed that, while they had been duly taken the night before, they had not yet been interpreted. He first expressed his regret at the popular disbelief in them, and, declaring ignorance to be the cause, sum-

moned me to explain them to the people. After a general account of augury, its varieties and significance, he turned to the officiating augur, described what he had seen in his *tabernaculum* the previous midnight, and asked an interpretation, which turned out to be favorable. At this point, however, impromptu thunder and lightning were quite unexpectedly introduced, and, as thunder was always unfavorable and the lightning was on the right, an immediate postponement of the assembly till the next day seemed imminent. The claims of the gods were satisfied, however, by assuming it to be the next day when the *comitia centuriata* actually organized an hour later.

The prayer and sacrifice, with which the *contio* regularly opened, we were obliged to omit on this occasion, but they will be added on the next. The consul proceeded to explain the purpose of the *contio*, and, after enumerating the offices to be filled by the succeeding *comitia*, appealed to all citizens to exercise the suffrage wisely. He then called upon representatives of the various parties, and each in a brief Latin speech presented the claims of his party and urged the citizens to vote for his candidate for consul. With the phrase, "impero qua convenit ad comitia centuriata," the consul directed the citizens to arrange themselves by centuries, and the non-citizens (i. e., non-Latin students) to withdraw. The *contio* then became theoretically a *comitia centuriata;* but obedience to this command was temporarily postponed, and the assembly without any change became a second *contio*, preceding the *comitia tributa* and presided over by a prætor, who now entered the hall attended by his two lictors, and with similar formalities summoned the *comitia tributa* immediately to assemble and elect the officers enumerated. At the conclusion of this *contio* the assembly was supposed to rearrange itself by tribes, but the actual procedure was again postponed, and it becomes a third *contio* preceding the *concilium plebis*. The *tribunus plebis*, selected by *comparatio*, enters unattended, and, with much less ceremony (the omission of the *auspices* may be especially noted), directed the patricians to withdraw and the plebeians to arrange themselves by tribes for the election of the plebeian officials. But before this was carried out, a fourth *contio* was held, presided over by the *pontifex maximus*, who summoned the *comitia tributa* and announced that he would select by lot 17 of the 35 tribes to vote for

the purely religious officials. When the *pontifex maximus* had pronounced the official formula, the four presidents with their lictors withdrew, and the four *contiones* (after an interval during which half the regular school session was held) thereupon successively arranged themselves, at the summons of *praetores*, into the four assemblies, each one of which was supposed, of course, to have followed directly its corresponding *contio*, directly the formula was pronounced. The *comitia centuriata* assembled first, arranged by its classes and centuries. The 70 *centuriones* of the *prima classis* drew lots from the consul to select the *centuria praerogativa*, which had the honor of announcing its vote first, and the ballot of this *centuria* was read to the assembly. The ballots were then collected in the order of classes by the consul and his assistants, the *custodes* and *diribitores*. No attempt was made on this occasion to represent the *saepta* and *pontes*.

The prætor at this point took charge and, at a signal from him, the assembly organized as a *comitia tributa*, with just six minutes needed for the change. The 35 *curatores* were then summoned to draw lots to select the *principium*, a distinction corresponding among the tribes to the *centuria praerogativa* among the centuries. The lot fell on the Falerna tribe, which thus had the privilege of announcing its vote first. Each tribe was then assigned to a classroom, corresponding to the ancient *saeptum*, to which it was to withdraw for the casting and counting of its ballots. The actual withdrawal was naturally postponed until after the other assemblies.

The next president (for reasons noted below) was the *pontifex maximus*, who summoned the 35 *curatores tribuum* to draw lots again, to determine what 17 tribes should vote for the religious officials. When this was decided, the tribes were, as before, directed to repair to their respective *saepta* and vote simultaneously.

Finally the *tribunus plebis* took the chair, and at his command all patricians withdrew, repairing to the tribal *saepta* previously assigned. The *concilium plebis* was held last, in order that this important act might be immediately carried out. The 35 *curatores* once more drew lots for the *principium*, and then all the tribes withdrew to their rooms, where each tribe, as it cast its ballots, represented successfully a unit of the *comitia tributa*, the religious assembly, and the *concilium plebis*.

The counting of the ballots was itself a problem. Each tribe, for example, cast its ballots for quæstor, and after finding what 20 candidates had carried that tribe (i. e., had received the largest number of votes), the list was sent to a central room. Here it was joined by lists from the other 34 tribes, and it was then decided what 20 candidates had carried the largest number of tribes. These were declared elected. The same procedure was followed for all the officials except consuls, censors, and prætors. Here the problem was to decide what candidates had carried the largest number of the 373 centuries. The ballots, which had been marked with the name of the tribe and century, and had been collected by classes, were arranged into the 18 centuries of *equites*, the 70 centuries of the *prima classis*, etc., down to the 5 extra centuries of *capite censi*. The vote of each century was then determined, often by lot, and 373 ballots were drawn up for the final count. It may be noted here that since the election the censor has so arranged the assembly that each century is organized as a unit — i. e., in each *classis* the *iuniores* and *seniores* are separated and arranged within themselves by tribes alphabetically. Thus all the *iuniores* of the *tertia classis, Aemilia* tribe, are seated together, have elected a centurion, and at the next election will make out one ballot for *consules*, *censores*, and *praetores*. This will produce directly the 373 ballots, and enable us to complete the count in half an hour.

The general result of the election was a victory for the *Populares*, and the consuls of our Roman state for the next year are Lowenthal and Boyd; the censors, Barss and Miss Bickford; the *pontifex maximus*, Miss Alden — a list indicating one important reform in the suffrage of ancient Rome.

Shortly after the election, the several officials who presided over the electoral assemblies made their respective *renuntiationes*, in which they read the list of successful candidates. Each presiding officer illustrated his power of rejection by refusing to confirm the election of one candidate. Lest an actual rejection might give offense, an invented personage with a Latin name was inserted in the lists, upon the rejection of whom the actually successful candidates were declared elected.

Somewhat later the *comitia curiata* assembled and, under the pres-

idency of the preceding consul, passed the *Lex de Imperio*, conferring the *imperium*, or right to exercise the powers of the office, upon the consuls and prætors, the only officials possessing it.

No ceremonies attended the entrance into office this year. Next year it is expected that the inauguration of the newly elected officials, together with the formal *abdicatio* of the old, will be properly observed.

The election over and the officials installed, the normal life of the state should begin. It is true that for the first two years the life of the state consisted solely of the election and the preparations for it. But with this year began a new era. While the development will naturally be slow, it will continue until the election and campaign have assumed their normal relation. The administration is conducted by a body of officials, the idea of which is borrowed from the Empire, called the *Publicum Consilium*, and composed of the two consuls, the two censors, the *pontifex maximus*, and one from each of the other bodies of officials. This body meets on one stated evening each month, and at such other times as the consul that has the *imperium* for the month decides. The consuls alternate monthly in the exercise of the *imperium*, although, of course, both possess it continuously. Through this *consilium* the various officials are assigned their duties, and to it they are immediately responsible. The consuls have general jurisdiction and supervision of all the activities of the state. The censors, as has already been noted, are engaged in reorganizing the *com. cent.* on a permanent basis, and are furthermore revising the list of patricians. The number of these has been reduced to sixty, and the honor of patricianship, subject to revision each term, is to be conferred upon the best Latin scholars, provided the average of their other studies equals their Latin. A visible mark of this honor is to be conferred upon these patricians shortly in the name of the state, in the form of a pin or medal, for which the censors are endeavoring to secure an original and characteristically Roman form, by offering a prize to that citizen who offers the best design. "Hoc modo," they state in their edict, "hanc artem humanissimam ac liberalissimam in nostra civitate fovebimus."

The first duty of the *pontifex maximus* was to make a complete record of all successful candidates, and secure such information about them as would be necessary — the year, address, rooms, etc.

She has also been devising some means of representing the *Fasti Consulares*, and a frieze in burnt wood will shortly adorn the walls of a room selected as the *Regia*, inscribed with the names of the chief officials elected each year. Under the general direction of the *pontifex*, the *viri epulones* are preparing for a Roman banquet to be made an annual custom of the graduating Latin students, at which, so far as practical, the ancient customs will be imitated. She has organized some of the appointive religious bodies, notably the *Vestales*, the *Fratres Arvales*, the *Flamines;* and their installation took place recently with interesting ceremonies. The *XV viri sacris faciundis* are preparing, under the same supervision, a body of Latin songs, which will be issued in permanent form, to be taught to the pupils and sung in assembly.

The managers of the *Equestres* party recently took occasion to present in two-minute speeches before all the classes a severe arraignment of what they termed the do-nothing policy of the *Populares* administration. The latter made answer a day or two later, and the outcome of the controversy is a charge of conspiracy brought against one of the *Populares* leaders. The prætors, who had in the meantime been assigned their respective judicial functions by lot, are preparing for a public trial under the direction of the *praetor urbanus*. The jury system is being worked out practically, the methods of procedure, etc.; while the expected conviction of the accused will be followed by an appeal to the people. The effort is being made to produce conditions resembling those of the Catilinarian conspiracy as far as possible.

The *Aediles* are considering the presentation of a Latin play next year, and making preliminary investigations in which they will gladly welcome any information from institutions where plays have been presented. They have charge also of all property belonging to the state, and provide for whatever insignia are needed, such as the *fasces*, etc.

The *tribunus plebis* has made such changes in the arrangement of the tribal assemblies as to make the location of each tribe permanent hereafter, and has general supervision of the *curatores* and *tribuni aerarii*. It was naturally the *concilium plebis*, presided over by this officer, that voted to confer decorations on the patricians and levied a *tributum* upon the tribes through the *tribuni aerarii*.

The *quaestores*, who are divided into *quaestores urbani Italici* and *militares*, assist the censors in all their duties and take charge of the funds collected by the *tribuni aerarii*.

In a word, without permitting any one of the 150 officials to assume burdens that would interfere with regular school work, we have attempted to assign to all some characteristic duties that would make clear to them their position in the state.

It may be noted in general that, while the present year represents 64 B. C., the effort has been made to present each assembly, official, and function in its most characteristic form, no matter at what period it possessed that form. This has led to some minor chronological inconsistencies, which are, however, more valuable than would be a strict adherenee to a specified period, when the forms appearing would often be unintelligible. We intend ultimately to select four dates in the last century of the Republic, when the election was especially critical—such as 133, 91, 64, and 50—and thus secure a definite historical setting.

It will be unnecessary to restate here the valuable results I am confident are derived from the inauguration of our Roman state. Its complete success here has been due to the presence of three indispensable conditions, the absence of any one of which would be fatal: absolute faith in his project in whosoever engineers it, sufficient to insure persistence until results appear; the hearty support of school authorities; and, most essential of all, a corps of enthusiastic Latin instructors, to whose professional spirit and hearty co-operation is due whatever of success our Roman state has thus far achieved.

IS MANUAL TRAINING AS NOW TAUGHT EDUCATIVE?

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Comparatively speaking the most uneducative, the most unpedagogic, the most unreasonable subject taught in the American elementary school today is hand-work, in any of its forms, whether it be what is known as manual training, domestic art, or domestic science. Observe whatever recitation you please, and you will have the privilege of witnessing one which to a large extent is void of real thought, void of pedagogical principles, void of reason upon the part of the pupils, but abounding in dogmatism, disconnected facts; with an abundance of "doing," but with no conscious theory in the doing.

If your visit is to a manual-training room, you will likely see a teacher giving a lot of hasty directions as to how the thing is to be done; then you will see the pupils "blaze away" trying to "do what she said to do." And that is all that you will see, unless it be that the teacher attempts to give some personal attention to see if the rules given are being followed to the letter. This may be woodwork, but it certainly isn't brain-work.

Let your visit be to a class in sewing, and you will see a teacher reading directions from her notebook, (which contains *verbatim* the directions copied from a good design-book purchased for twenty-five cents), to the class about gores, stitches, seams, designs, etc. The class copies and follows orders. Orders having been executed the class is dismissed. There is plenty of hand-work in this recitation, but little head-work.

One certainly would expect better results from a visit to a class in cooking; for perhaps the splendid amount of rich thought back of all hand-work is more prominent here than elsewhere. All one will see, however, is a teacher well versed in Mrs. Rorer's latest cookbook, giving directions, and a slavish following of these directions upon the part of the pupil. Even the immense notebook kept by the pupils, in which they record each recipe learned, does not relieve the ab-

sence of reason and the dogmatic subservience to rules which have dominated the work of the hour. Here again is found plenty of "learning to do by doing," but little "learning to think by thinking."

There could scarcely be a better illustration of mental slavery than this. The child's reasoning faculties are not seriously applied in the process of learning. He must simply follow slavishly the rules given by the teacher. The whole subject rests therefore upon his circumstantial memory. If he works long enough to fix in his mind the different details of each subject, and the different subjects themselves, then he may retain them; but, if not, it all ends in a jumble. So far as his reasoning faculties as aids to his memory are concerned, they are not appealed to by the teacher. The apprentice system is quite as educative to the pupil as is this kind of instruction given in the majority of manual-training recitations. In each case he is not given a large principle to reason out in its details, but is simply left to follow in the most servile manner such directions as are given him. This absence of reason and thinking is clearly manifest in the attitude of the teacher, who has the feeling that the recitation has not begun until the tools and materials to be worked with have been picked up, and that it is finished so soon as the tools are laid aside and the product is finished. The training of the pupil is not the chief idea in this process. Until the teacher has thought in mind and succeeds in getting that thought into the child's mind, all hand-work amounts to nothing but a mere handling of tools and materials. Hand-work is not properly educative until the doing upon the part of the teacher and pupils is the conscious application of a clearly conceived theory. So long as it is conducted upon any other basis than this, the teacher is nothing more than a carpenter, seamstress, or cook. It is conducted upon this trade idea almost universally at present; and this trade notion certainly is not educative. It would be a very difficult task for the most enthusiastic manual-trainingist to show any material difference between this kind of teaching and that given by the slave-owner to his negro servants in the days of slavery. If the master desired a carpenter or a blacksmith, he simply apprenticed his negro to one skilled in the art, who taught him how to do the thing by doing, who gave him directions and showed him how to carry them out. The mistress read to her colored ser-

vant from a cookbook how to make each different dish, then showed how the directions were executed; the illiterate negress followed orders, and in the course of years became the best cook in the neighborhood. The same method was followed in teaching the colored women to sew. Thus there were manual training, domestic science, and domestic art taught by one who was not a professed teacher, with the same methods which are followed today by one who claims to be an experienced educator. What is more deplorable, we are getting not one whit better results than did the slave-owner from his teaching, but rather we are getting the same results. If we do get anything else, it is not because of our teaching, but because of the advancement in science. In both the industrial school, as such only, and the slave school the results and aims were the same, viz., the finished product and the skilled laborer. By skill each implied the laborer who could turn out the best product with the greatest speed.

We claim that this attitude and method are uneducative and unpedagogic, because they lay no stress upon the rich field of thought represented by each one of the departments mentioned above. They do not lead the pupil to sympathize with the great fields of real life from which they are taken. He may be thereby prepared for doing a certain thing, but is not prepared for life at all. In fact, all this kind of teaching does not look so much to the growth of the learner and the needs of society, but to the skill in rapidity and the production of a finished article. The pupils thus trained never do much more than the actual performances taught them in the school, for their power to reason in general or in particular is not trained at all.

What have been the effects of manual training upon the patrons of the school? They accepted it from its advocates at first because it was said to be a "practical" subject, and one that taught the pupils how to "do" something that was useful. But most of the plain-thinking citizens, with a fair amount of power for observing things, have concluded that seventy-five cents will purchase all the finished products which a schoolboy turns out within a year. The mother has discovered that in two days she can teach her daughter to cook as many real dishes which are used in her family as she is learning in two years at school. Both parents have concluded that there is quite a deal of time wasted by this so-called hand-work. It is, there-

fore, only a matter of time when the subject will be doomed to give place to the demands of numerous other subjects which can show sounder reasons for their claims to a place in the curriculum. Those of us who believe that hand-work must not perish from the curriculum must justify its retention upon sounder principles. We believe that there is an abundance of thought in hand-work. We believe that there is ample reason why this subject should be taught. We think it can be made highly educative. We think it is imperative that it be made so. It will be the purpose of the article following this to suggest at least some principles by which we are to be guided in discovering "how hand-work may become educative."

THE SOCIAL SCIENCES IN HIGH SCHOOLS

H. A. HOLLISTER
University of Illinois

Considerable interest is manifested of late, especially in the West, as to what is the best content of courses to be offered in history, civics, and economics. Ever since the report of the Committee of Seven on the teaching of history was promulgated, there has been a decided tendency to put its recommendations to the test, and as a result the history work in high schools has come to have a certain uniformity of type as to content, based upon these recommendations. Just at present, however, there is a restlessness along certain lines. There are those who feel that not enough time is found under the present plan for the application of the more practical phases of the social sciences, as in the teaching of civics and economics.

To an observer who is not a specialist in this particular department, but rather interested in the broader problems of the curriculum of secondary schools, the disappointment as to results seems due rather to the failure to accomplish that part of the work assigned to the elementary school, than to any inherent weakness in the work suggested for the high school.

There is a certain amount of concrete, basic study of institutional life which naturally belongs to the elementary period of instruction; but as the majority of our elementary schools are now organized, this work is impracticable. The failure of its accomplishment destroys the natural order of development, and throws upon the high school the necessity of dwelling too long upon rudimentary principles to be able to accomplish with any degree of satisfaction the work contemplated by the report of the committee.

At two rather important meetings in Illinois the past month—the Schoolmasters' Club and the High School Conference of the State University—considerable emphasis was given to discussion in favor of more formal teaching of economics and civics in high schools. Now, it seems to the writer that, while there are certain very good grounds

for such an argument, there is a better and more logical method of attaining what seems to be the desired result from our teaching of the social sciences.

There are a number of subjects, partly secondary and partly collegiate, which we may call phases of the philosophy of history, or art in history. Among these are literature, the science of government, economic doctrines, industrial development, various social problems.

Now, these subjects all take their significance chiefly from their historic background, and their rudiments are therefore to be found in a study of history. To attempt to teach them without a fair knowledge of the historic groundwork is an illogical and somewhat dangerous procedure. It exposes the student to the possibility of inheriting from his school training the same weakness which would result from so-called scientific reasoning without sufficient or accurate data. He would have the form without the substance.

For this reason the safer plan would seem to be to keep constantly before the student the historic groundwork. Probably no better field of historic work can be found for such a purpose than three courses of a year each in ancient, English, and American history.

I am not unaware of the argument favoring the shortening of the ancient history to a half-year. This is simply an effort to gain time for instruction in the various social and economic deductions from history. But if there is any good ground for what has been said in favor of history as the basis of such studies, this argument is not well grounded.

In the first place, the ancient nations—Greece and Rome, for instance—are the only historic nations whose cycles stand completely in perspective. Their theories of government, their socal conditions, their industrial systems, stand complete, both as cause and effect, so far as the life-history of these nations is concerned. They have also an art and literature and philosophy which are too fundamentally related to our present to be passed lightly by.

To give less than a year each to English and American history is to confine these practically to mere skeleton outlines, with scant opportunity of tracing the relation of historic movement to the development of a nation's laws, social life, art, and philosophy.

I know the argument is that such a dependence upon history leads

to a lack of definiteness in the training of the youth along the more practical lines, as it is put. This is assuming, of course, that there is no other remedy for such a lack of definiteness than the use of regular, distinct textbook courses in literary history, civics, economics, etc. This, however, is purely a habit of thought.

It is quite as feasible a plan—and much more logical, as I see it—to dissect out regular courses along these and other practical lines. I mean a regularly prepared, printed outline course, with its proper historic setting preserved. Such a course in economics, for instance, might run through the entire three years of study. This might also be true of literature. Civics, while necessarily confined mostly to the United States, might well have a beginning in connection with English history. These dissected-out courses, carried along parallel with the general historic study, and supported by numerous handbooks and source material in the reference library, would form a chain of continuity in the historic teaching of our schools, and would thus strengthen the influence of its lessons as well as the validity of the student's deductions.

Such an outline course, with references, would be to the text in history what the laboratory manual is to physics or chemistry. It would furnish sufficient opportunity for the application of a historic principle to fix the significance indelibly in the learner's mind.

A course thus pursued in the history of English literature, for instance, would economize time, and would also give to the student that knowledge of historic factors in the development of literary types which is so often entirely lacking in students who have had the history of literature without the English history.

EDITORIAL NOTES

THE NORTH CENTRAL ASSOCIATION

The eleventh annual meeting of the North Central Association of Colleges and Secondary Schools was held in the Auditorium Hotel, Chicago, Friday and Saturday, March 23 and 24. The attendance was representative and the program unusually interesting. Inasmuch as the association publishes its own proceedings, the *School Review* limits itself to a brief account of the meeting.

The first session of the meeting was held on Friday morning. After a gracious address by the president of the association, President George E. MacLean, of the University of Iowa, reports were made by the treasurer, the executive committee, the committee on athletics, and the delegate to the College Entrance Examination Board. The paper of the session was presented by Principal George W. Benton, of the Shortridge High School, Indianapolis, answering the question, "What the Colleges Can Do to Assist the High Schools in More Nearly Satisfying the Demands of the Public." The lateness of the hour prevented the paper from receiving the discussion it deserved.

The program of the afternoon session consisted of a discussion, a paper, and two reports. The discussion was on the topic: "Are State Boards of Medical Examiners Justified in Excluding from Examinations the Graduates of Medical Colleges That Allow Advanced Credit for Work Done in a College of Liberal Arts?" The leaders of the discussion were Dr. Frank Billings, of the University of Chicago, and Dean J. H. T. Main, of Iowa College. While the topic is one that concerns the relation of the colleges to the professional schools, it may be of interest to note that the general tendency of the discussion was toward the recognition of work wherever done, provided the quality and quantity were unquestioned. Director George N. Carman, of the Lewis Institute, presented a paper on "The Inspection and Accrediting of Colleges and Universities." The accrediting of schools is a subject that has received careful consideration from the association. Plans and methods for doing this important work have been adopted and put into operation by the association. The commission on accredited schools has found it impracticable to stop with the consideration of the schools, but has been compelled to consider the character of the colleges as well. A large number of subjects required by the colleges for graduation are offered both in the college and in the school, the college accepting the work, when well done, from either institution. The improvement in instruction and equipment in the better secondary schools has brought these schools to a point where they may well ask whether all colleges in the territory are doing equally good work in the subjects taught by both. It is but fair, therefore, to raise the

question of the advisability of extending the work of the commission to include the inspection and accrediting of colleges. The association referred the question raised by the paper to the commission for further consideration and report.

The session of Saturday morning was devoted to a discussion of the subject: "To What Extent and by What Methods Should First-Year Students in Colleges and Secondary Schools be Supervised?" Dean Thomas A. Clark, of the University of Illinois, presented a thoroughgoing and helpful paper on the subject, and was followed by Principal George E. Marshall, of the Sioux City High School, Chancellor Winfield S. Chaplin, of Washington University, St. Louis, and Principal Frank B. Pearson, of the East High School, Columbus. All the speakers were of the opinion that these critical first-years should receive increased attention from the colleges and schools. All emphasized the need of the personal element in dealing with the students, and recognized how rare the man is who has both the talent for and interest in this important work. An important consideration, hardly raised in the discussion, is how far the organization and method of education in the lower schools are responsible for the inability of the students to meet the critical tests of change from one school to another.

With the election of officers, the program of the meeting was brought to a successful close.

THE HARVARD DEPARTMENT OF EDUCATION

By a recent vote of the Faculty of Arts and Sciences of Harvard University, the instructors in education have been constituted a "Division of Education" of co-ordinate rank with other divisions of the faculty. Heretofore the instructors in education have been affiliated with the Division of Philosophy. Besides assigning to education its appropriate place among other university studies, this vote evidently indicates that Harvard University intends to develop the study of education and the professional training of college-bred teachers as rapidly and as comprehensively as its resources permit.

MEETING OF THE N. E. A. POSTPONED

Just as we go to press we are notified that the Executive Committee of the National Educational Association have decided that, on account of the disaster in San Francisco, no meeting of the association will be held this summer. The teachers of the country will welcome an early opportunity to see a new and splendid San Francisco.

BOOK REVIEWS.

Graphic Algebra for Secondary Schools. By H. B. NEWSON. Boston: Ginn & Co., 1905. Pp. 19.

This pamphlet is intended by the author to meet the growing demand for graphical methods and illustrations in connection with elementary algebra in secondary schools. A recent committee of the American Mathematical Society, in formulating standard college-entrance requirements in algebra, strongly recommended the use of graphical methods and illustrations, particularly in connection with the solution of equations. This little book is the author's interpretation of the committee's recommendations, both as to the kind and the amount of such graphical work. It may be used to supplement any textbook in algebra.

The subject is treated under the following headings: the point, the linear equation and its graph, simultaneous linear equations, the quadratic equation, quadratic equations in two variables, and simultaneous quadratic equations.

The subject-matter of the book is clearly expressed, and would not be difficult of comprehension by the average student. In the exercises, which are well planned, the pupil is directed to construct the graphs of the different equations, and then verify his constructions by solving the equations algebraically.

The author has done his work admirably, and the pamphlet would be a valuable and interesting supplement to the ordinary high-school algebra. It is unnecessary to dwell upon the value of graph in vitalizing the study of equations, in relation to the study of science in the high school, and as a basis for advanced work in mathematics.

A. F. AMES.

PUBLIC SCHOOLS,
Riverside, Ill.

Advanced Civics: The Spirit, the Form, and the Functions of the American Government. By S. E. FORMAN. New York: Century Co., 1905. Pp. xxi+456.

Within the last five years several excellent textbooks on the subject of civics have appeared. Among these, *Advanced Civics*, recently published by the Century Co., merits special attention. In its clear, concise statement of facts, in its wise selection of the things most essential, and in its logical arrangement of subject-matter this book is almost without a rival. A great quantity of material, which, however important it might seem to the teacher, he has been compelled to pass over in his class work because of the limited time at his disposal, has been gathered up by the author of this book, and by some magic art packed away so neatly within its 450 pages that there is little appearance of crowding, and no uneasy sense that much of importance has been left out. Whatever he was unable to carry along in the text as baggage he has carefully boxed up in his "Suggestive Questions and Exercises," to be brought on as freight as time and circumstances might permit.

But while this book is a veritable encyclopedia of facts about our government, it is also a connected and interesting history of our political growth and a prophecy as to our future. The secret of this is found in the fact that the author views the American government as a living, growing organism. He does not treat his facts as a mass of dead, inorganic matter to be cast together in a fixed mold and worshiped as a pagan idol, nor as a lot of dry bones to be wired together with logical ingenuity into a repulsive skeleton, fit only to command the interest of political antiquarians. His facts are related to living organism. The American government is flesh and blood, as well as bones. It is the embodiment, also, of an immortal spirit — the spirit of a Simon de Montfort and a Cromwell, of a Hooker and an Otis, of Washington, Webster, Clay, and Lincoln; the spirit of thousands of patriots and statesmen who have toiled and sacrificed, who have gladly given up time and money, and even life itself, to hand down to coming generations civil and religious liberty as the common birthright of all.

The book is divided into three parts. In Part I we trace the growth of this spirit, revealing itself in charters and constitutions, in new laws, and in changing political customs from the day of Magna Charta down to our own time. This part of the book treats also of the hygiene of the body politic, as appears, for instance, in the following sermonette on the duties of citizenship: "The duties of citizenship are always equal to its rights. If I can hold a man to his contracts, I ought to pay my own debts; if I may worship as I please, I ought to refrain from persecuting another on account of his religion; if my own property is held sacred, I ought to regard the property of another man as sacred; if the government deals fairly with me and does not oppress me, I ought to deal fairly with it and refuse to cheat it; if I am allowed freedom of speech, I ought not to abuse the privilege; if I have a right to be tried by a jury, I ought to respond when I am summoned to serve as a juror; if I have a right to my good name and reputation, I ought not to slander my neighbor; if government shields me from injury, I ought to be ready to take up arms in its defense."

In Part II we study the anatomy of the American state as it now is: its framework in national and state constitution; its nerves and sinews and muscles in the organization and practical workings of political parties. The chapter on "Territories and Dependencies" and that on "Party Organization" in this portion of the book are of especial interest and to the point.

Part III deals with the functions, the physiology, of government. It does not stop short with a bare enumeration and classification of the services which government actually renders. It raises the serious question: "What functions ought government to attempt to perform?" In his discussion of this question the author aims to be fair to the different views held on this subject as typified by the individualist, the liberal, and the socialist. His attitude is that of a student and not of an advocate. He is a seeker after truth, and not a purveyor of cut and dried theories. This discussion of the functions of government offers, also, a fine opportunity for an introduction to the study of many vital and economic problems which the author has not neglected.

It was evidently the intention of the writer that his book should serve, primarily, as a textbook in civics in the more advanced secondary schools and colleges. It is well adapted to this purpose. But it is also a valuable handbook for every American citizen, an interesting guide into the field of politics, and an inspiring counselor to duty.

EDWARD E. HILL.

HYDE PARK HIGH SCHOOL,
Chicago, Ill.

A Text-Book of Physics. By G. A. WENTWORTH AND G. A. HILL. New York: Ginn & Co., 1905. Pp. 480. $1.15.

The advent of a new textbook of physics, or the appearance of a new edition of an old text, is greeted by every alert teacher of the subject with the hope that at last the much-longed-for ideal text has been written. The frontispiece of the revised edition of *A Text-Book of Physics*, by Wentworth and Hill, stimulates this aroused hope by the suggestion that the subject is to be vitalized by correlating theory and practical application in a manner that is stimulating and interest-holding, and, at the same time, thoroughly educational and disciplinary. Yet, there is little that is essentially new from this point till chap. 9, an addition to the old book, is reached. True, there are a few minor changes here and there, some of them in the right direction — for example, the experiment to illustrate the laws of fusion (p. 106); but most of them are altogether trivial, as the changing the lines in a diagram — e. g., Fig. 232. Speaking broadly, there is a great hiatus in the improvement from the frontispiece to chap. 9, which is entitled "Some Modern Applications of Physics."

Here our hopes are again raised by the expectation of finding much that can be used for reference to enlist and hold enthusiastic attention. While this hope is not completely devoid of justification, the full possibilities of many of the subjects treated are not reached. The selection of topics is perhaps not to be criticised. The presentation and discussion of some of them are excellent — e. g., "The Water Tube Boiler," and "Artificial Refrigeration." But some are handled in a most perfunctory manner, and one is left wondering why they were inserted at all. For example, the description of the air-brake consists of the bare naming of the parts, leaving the student with no more real idea of the mechanism than he had before. Here and there are such exasperatingly loose statements as these: "The piston descends, forming a vacuum between it and the head of the cylinder" (p. 449); "The kilowatt hour . . . . means a current which will perform 1,000 watts of work per hour" (p. 450). The body of the text itself is not free from this defect. For example, a whole page (7) is spent in distinguishing between weight and mass; then on p. 9: "$\text{Density} = \frac{\text{weight or mass}}{\text{volume}}$"!

It had been expected that such errors would be eliminated; that the discussion from a mathematical point of view of certain highly interesting phenomena, to the utter beclouding of their physical meaning, would be avoided (e. g., the discussion of circular motion, p. 210); that the pedagogical fault of including matter — though eminently worthy in itself, and deserving of extended discussion and varied illustration, yet almost wholly unrelated to the rest of the subject — would be shunned (e. g., "The Laws of Chemical Combination," p. 166); that some reality would be given certain most important laws by ample citation of experiment (e. g., "The Laws of Fusion," p. 106, or "The Laws of Ebullition," p. 111); that there would be some obviously desirable rearrangement of the subject-matter. But we are disappointed, and turn from the book with the feeling that a splendid opportunity immensely to improve a text that has done and is doing good service has been unpardonably neglected.

B. O. HUTCHISON.

SHURTLEFF COLLEGE.

Advanced Arithmetic. By E. A. LYMAN. New York: American Book Co., 1905. Pp. 253.

This book is intended for pupils who have completed the grammar-school course in arithmetic. The author has been successful in his aim to develop fundamental principles, and at the same time to include the essentials of commercial arithmetic. The principles of arithmetic are carefully developed, the demonstrations are clearly put, and the problems throughout are practical and well chosen.

The author states that the exercises have been selected largely from actual business transactions. That is to say, the author seems to emphasize the disciplinary value of the subject, and at the same time apply the fundamental principles to the solution of practical business problems. Those of percentage, especially, bear out the author's claim that they have been selected from actual business transactions, or reviewed by business men. In view of this claim, however, it is disappointing not to find, in the earlier part of the book, a larger proportion of problems which represent actual contact at first hand with social, industrial, and business operations.

Constantly throughout the book the pupil is taught to check his work. It is admirable advice to give a pupil, to urge him to form the habit of estimating results before beginning the solution of a problem. It seems unfortunate, however, that the author, in common with the authors of other high-school arithmetics, should recommend the method of checking addition, subtraction, multiplication, and division by the casting out of 9's, instead of the much simpler and more economical method of reducing the different numbers to a single unit.

Attention should be called to the following errors that should be corrected in a future edition: On p. 53 we are told that $\frac{5}{8} \div \frac{3}{4} \times \frac{9}{7} = \frac{3}{4}\frac{5}{6}$ instead of $\frac{1}{1}\frac{5}{4}$. That is, $\frac{5}{8}$ is multiplied by $\frac{9}{7}$ and the product used as a divisor of $\frac{3}{4}$. Again, on p. 87 a problem is solved by means of an equation. We are told to check the result by substituting 90 for x in the original equation. This does not check the result. The substitution should be in the problem. The definition of geometrical progression (par. 326) is misleading.

Historical notes appear from time to time which add to the interest, and furnish valuable insight into the development of the subject.

Finally, this book is well adapted to furnish a practical course in business arithmetic to high-school and normal-school pupils, and deserves a careful examination by anyone selecting such a text book.

A. F. AMES.

PUBLIC SCHOOLS,
Riverside, Ill.

Elementary Algebra. By J. H. TANNER. New York: American Book Co., 1904. Pp. ix+364.

One only needs to read the preface of the present book to become impressed with the many excellent pedegogical points the author has sought to make in the preparation of the book. For example, an introductory chapter seeking to make the work in algebra seem to the pupil a continuous and natural outgrowth of arithmetic is particularly laudable. But this is only one of the numerous ways in which Mr. Tanner recognizes that an important element in the teaching problem is the taught.

From p. 32, just after the introduction and a very teachable chapter on positive and negative numbers, the work begins to take its bearings from the equation. On the whole, one must commend the rational procedure of this book, but the writer can

not longer resist raising the inquiry what advantage certain writers of elementary mathematical texts—Professor Tanner among others—hope to gain in point of logical, or any other sort of completeness, by the following definition of multiplication: "*The product of two numbers is the result obtained by performing upon the first of these numbers (the multiplicand) the same operation that must be performed upon the unit to obtain the second (the multiplier)*"?

Let us see: $5 \times 4 =$ what, on this definition? In this definition nothing is said about how the *unit* is to be "performed upon" to give the 4. I have a perfect right therefore to perform upon it thus: $1^3+1^2+1^1+1=4$, in accordance with the law x^3+x^2+x+1. Now I must "perform on" the 5 in the same way to get the product. This gives me $5^3+5^2+5^1+5=160$. Therefore, in the form stated 5×4 may just as well be 160 as any other number.

Of course, this definition is thoroughly "innocuous," because no high-school pupil ever understands it, and no high-school teacher ever uses it.

The writer would not have the reader believe for a moment that the character of Professor Tanner's book is in any way epitomized by this pointless attempt to introduce fuller adequacy into the definition of multiplication. Pretty much every writer who is impressed only with the need of logical perfection in the high school has used it, and some go so far as to call high-school teachers to account for not using it. The writer challenges it in Mr. Tanner's book, because the book possesses so many distinct points of merit that it can easily stand the challenge.

Mr. Tanner's treatment of factoring is excellent—much superior to the customary treatment. Type-forms typify something to the pupil as he treats them. The book contains many lists of problems, an unusually large number of which have a meaning and are worth solving by high-school boys and girls. Many of the problems are modern in a true sense.

The author evidently does not believe in the early use of graphs in algebra. The book contains nothing before p. 314 on the graph, and this, the writer believes, is unfortunate. Here there are three short sections on "Graphic Representation of Equations." A chapter on "Mathematical Induction" is a valuable feature. Aside from the rather formal development of topics, and a little too early and too continuous insistence on work by rule, the writer regards this as a good one. To say that it is one of the most teachable books of the Cornell Series is no mean praise.

G. W. MYERS.

UNIVERSITY OF CHICAGO.

BOOKS RECEIVED

(The notice here given does not preclude the publishing of a comprehensive review.)

EDUCATION

Principles of Teaching, Based on Psychology. By EDWARD L. THORNDYKE. New York: A. G. Seiler, 1906. Pp. xii+293.

HISTORY AND CIVICS

The Making of the American Nation. A History for Elementary Schools. By JACQUES WARDLAW REDWAY. New York: Silver, Burdett & Co., 1905. Pp. xii+412+56.